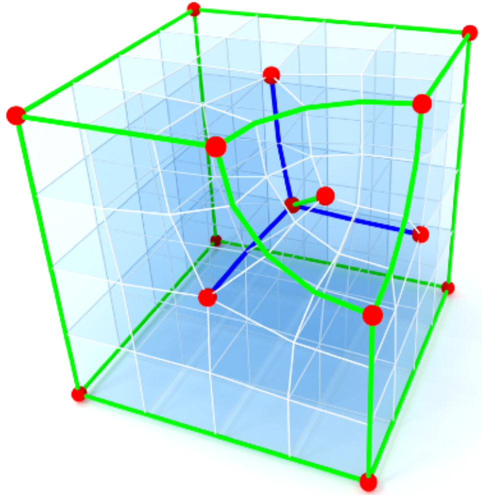
# Integer-Grid Maps

## for Hexahedral Mesh Generation

David Bommes
Computer Graphics Group



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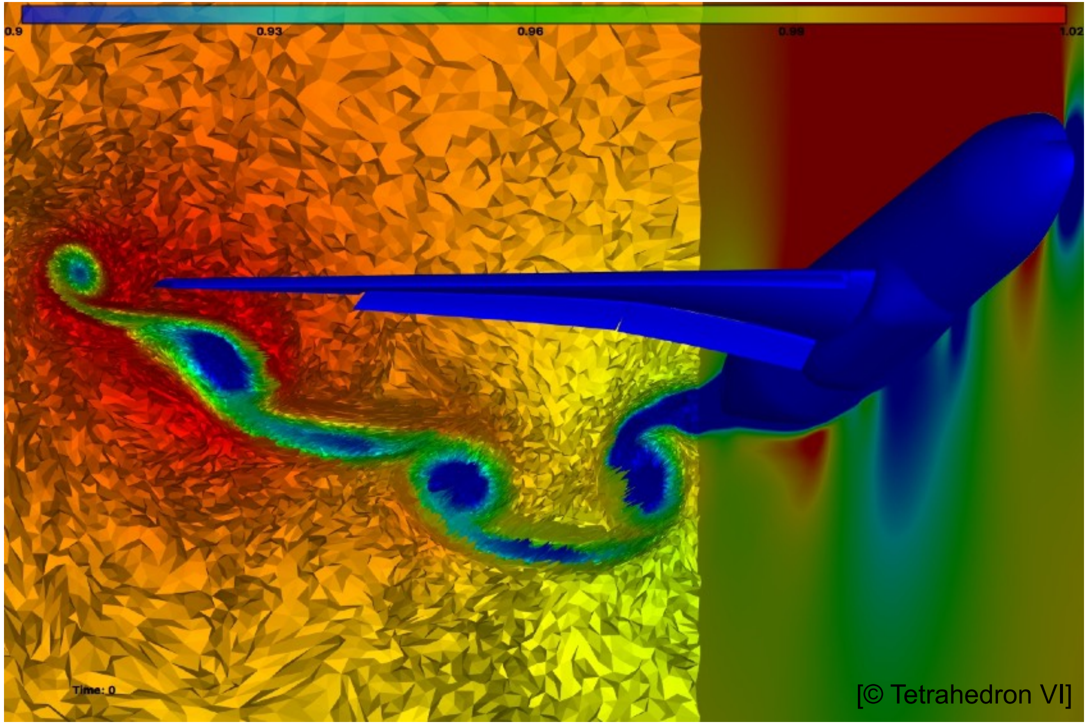
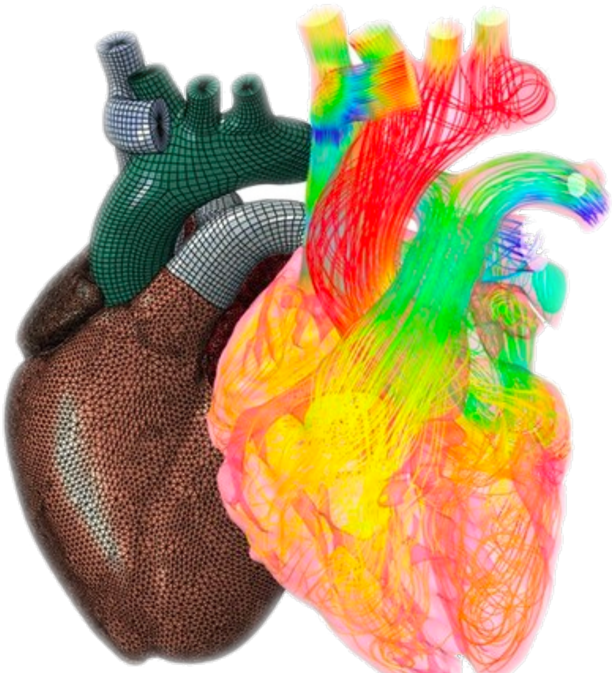
# Context

# Simulation depends on Volumetric Discretiztion

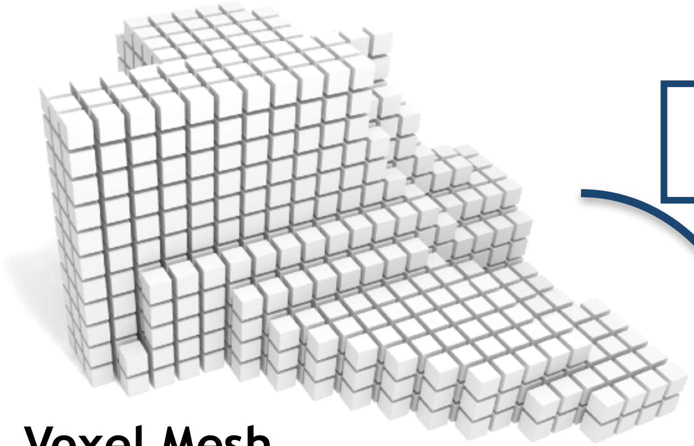



[© Tetrahedron VI]



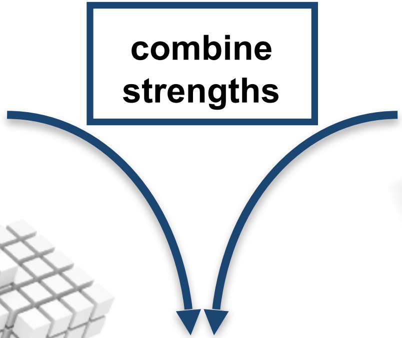
[© FlowVision]

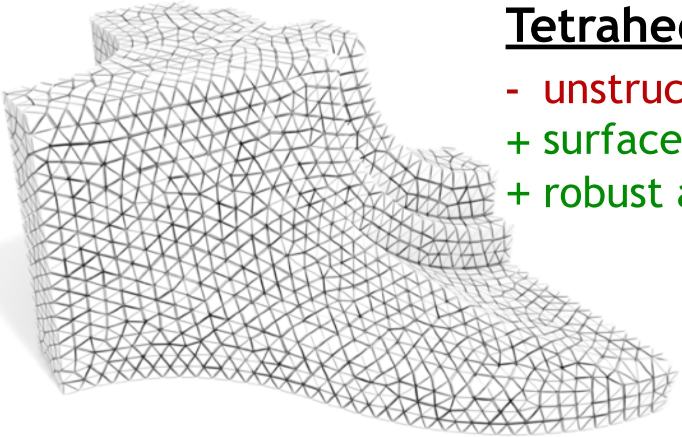
# How to discretize volumetric domains?



**Voxel Mesh**

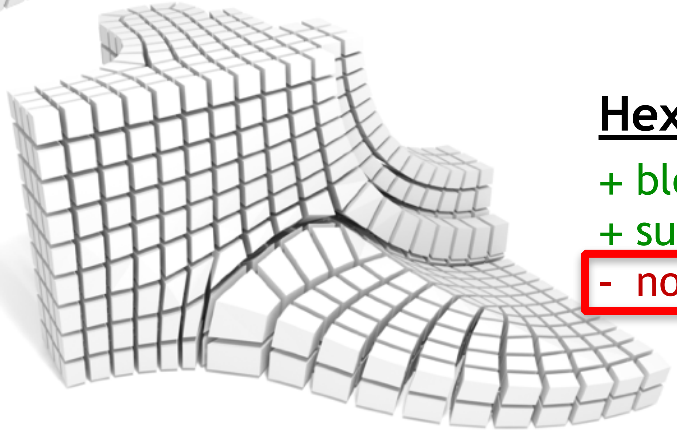
+ structured
- no surface alignment
+ trivial generation



**Tetrahedral Mesh**

- unstructured
+ surface alignment
+ robust algorithms



**Hexahedral Mesh**

+ block-structured
+ surface alignment
- no automatic algorithms

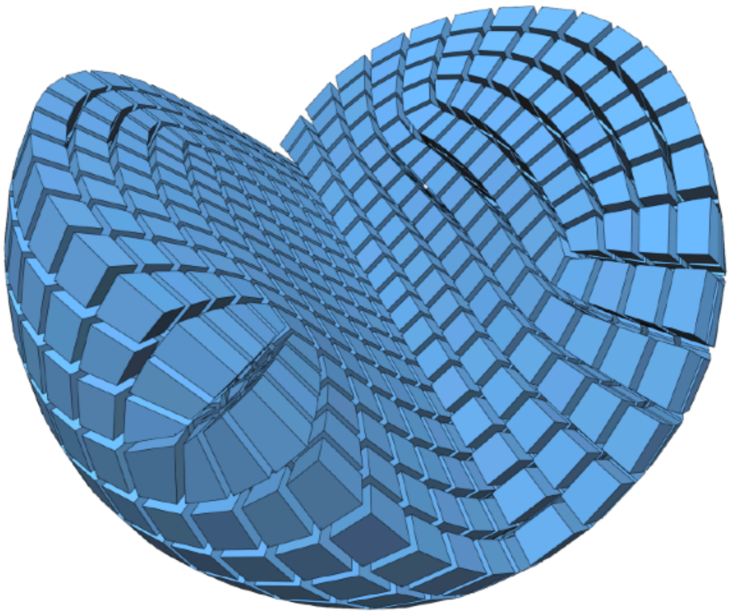
**Breakthrough required!**

# **HexMeshing**
# Challenges

# Why is Hexahedral Meshing so difficult?

- good **hexahedral meshes** require…

1. **approximation**
   - faithful boundaries & internal structures
2. **complexity**
   - low number of elements
3. **regularity**
   - few singularities/irregularities
4. **element quality**
   - low geometric distortion
5. **anisotropy and sizing**
   - application dependent



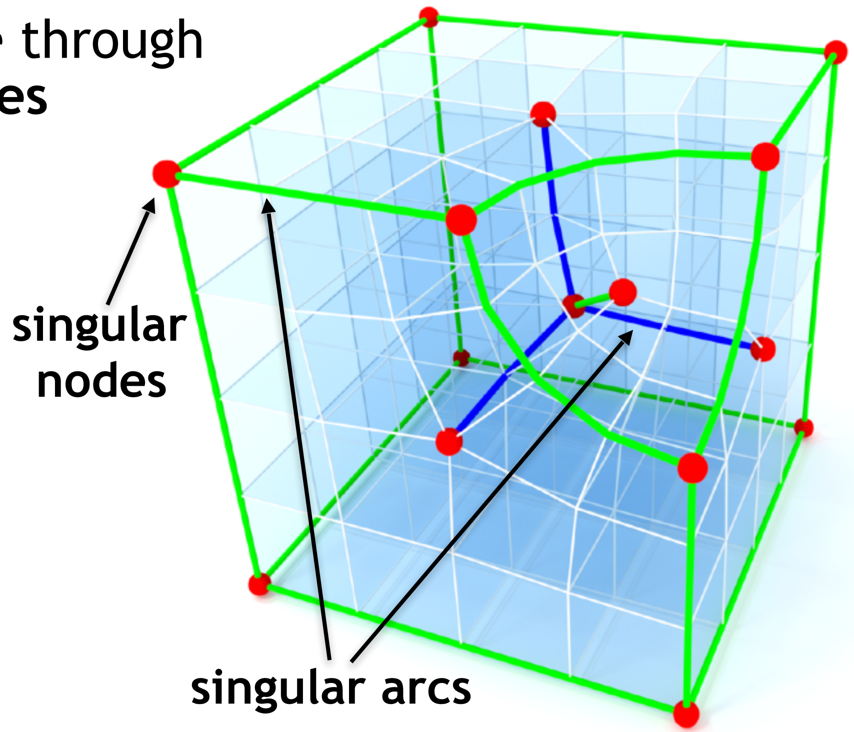
**conflicting objectives**

→ **existing algorithms optimize only subset**

→ **need holistic approach**

# Why is Hexahedral Meshing so difficult?

- **Key observation:**
  good block-structure only possible through
  **global optimization of singularities**




**State of the art:**
Manual decomposition into simple parts

# Integer-Grid Maps Approach

# A new approach for an old problem

**Main Principle:** View hex mesh generation as a **holistic** optimization problem and apply **global optimization**

## Simultaneously Optimize

1. approximation
2. complexity
3. regularity
4. element quality
5. anisotropy & sizing

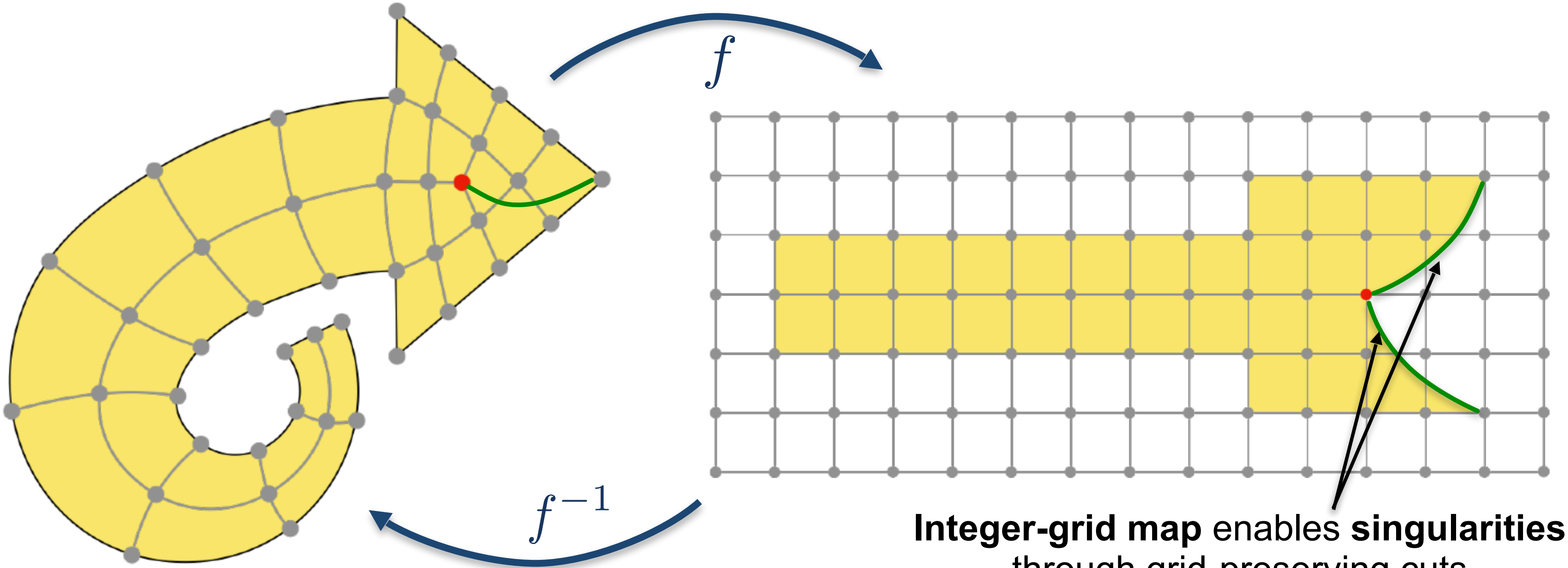
## Key to Success

1. Suitable parametrization of the problem
2. Scalable and **global** optimization strategy

**Problem:** naïve formulation gives practically infeasible large-scale non-convex mixed-integer problem

# Suitable Parametrization of the Problem

**Idea:** Interpret **Mesh Generation** as **Map Optimization**




**Integer-grid map** enables **singularities** through grid-preserving cuts

# Suitable Parametrization of the Problem

**Idea:** Interpret **Mesh Generation** as **Map Optimization**

## Optimize Mesh

1. approximation
2. complexity
3. regularity
4. element quality
5. anisotropy & sizing

**highly discrete**

→ translates into

## Optimize Map

1. alignment of map
2. grid volume
3. map distortion
4. map distortion
5. metric of map

→ translates into

## Variational

$$E(f) \rightarrow min$$

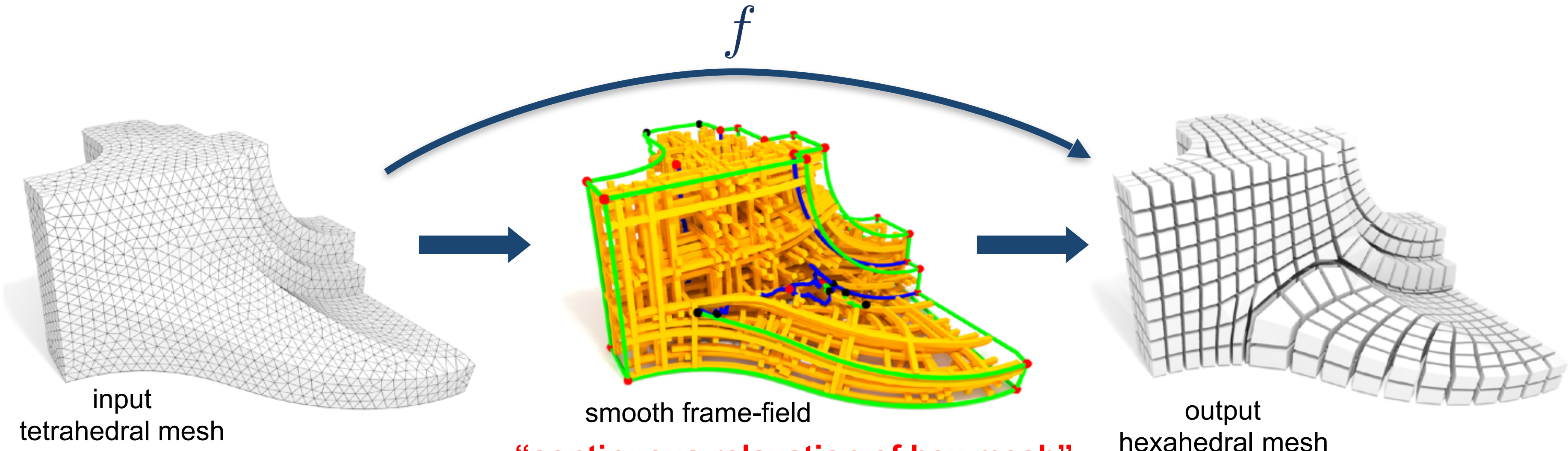
Mixed-Integer Problem

**less discrete since elements implicit in map**

# Scalable and Global Optimization Strategy

**Problem:** Mixed-Integer Problem $E(f) \rightarrow min$ **still too difficult**

**Idea:** Decouple into series of (geometrically motivated) relaxations



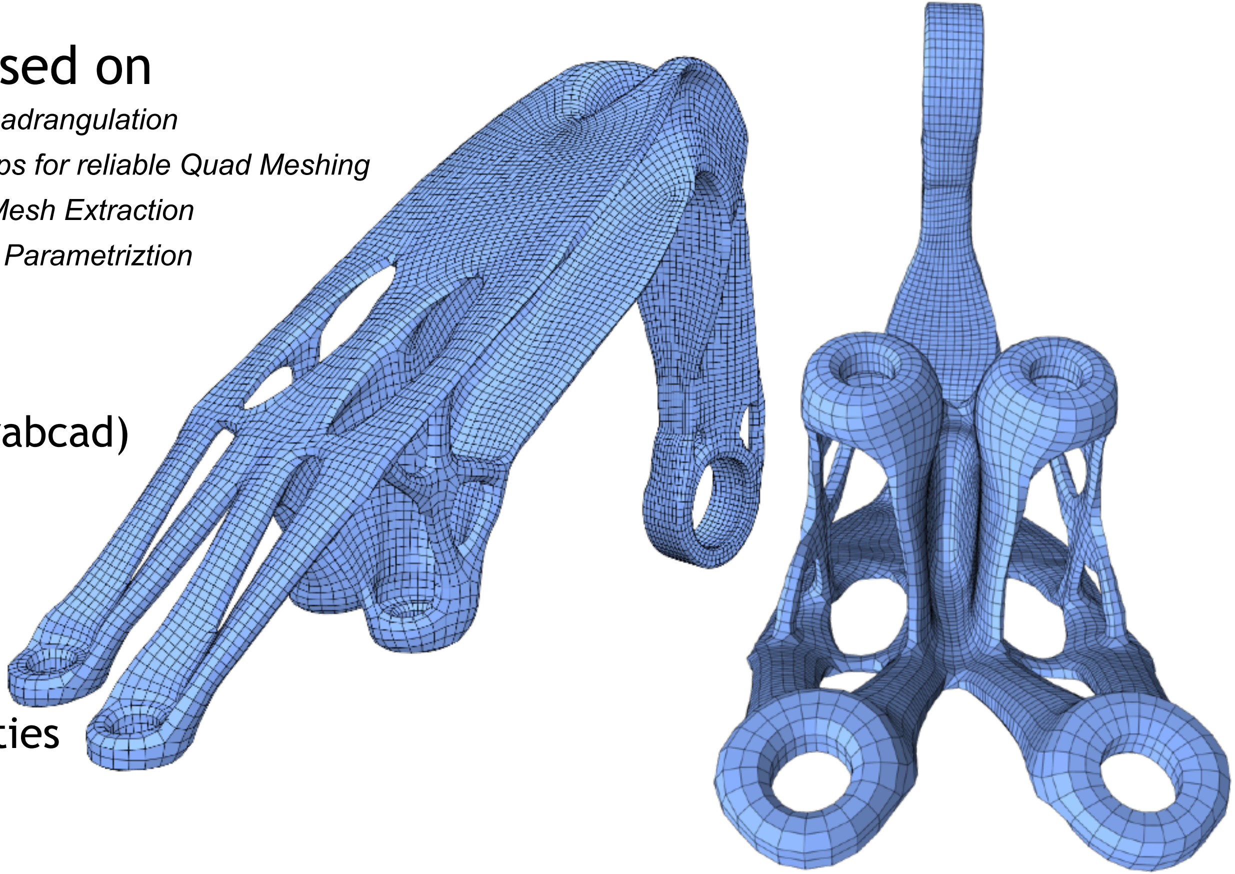

**"continuous relaxation of hex mesh"**
**→ even less discrete**

# **State of the Art**
# Quad Meshing with IGMs

# Success Story of Integer-Grid Maps in Quad Meshing

## **Technology Transfer** based on

- [Bommes et al. 2009] – *Mixed-Integer Quadrangulation*
- [Bommes et al. 2013a] – *Integer-Grid Maps for reliable Quad Meshing*
- [Ebke et al. 2013] – *QEX: Robust Quad Mesh Extraction*
- [Campen et al. 2015] – *Quantized Global Parametriztion*



## Example Model:

- airplane bearing bracket (grabcad)
- #triangles = 215k
- genus = 19

## Output:

- one-click solution
- globally optimized singularities
- runtime **80s** (#quads = 17k)

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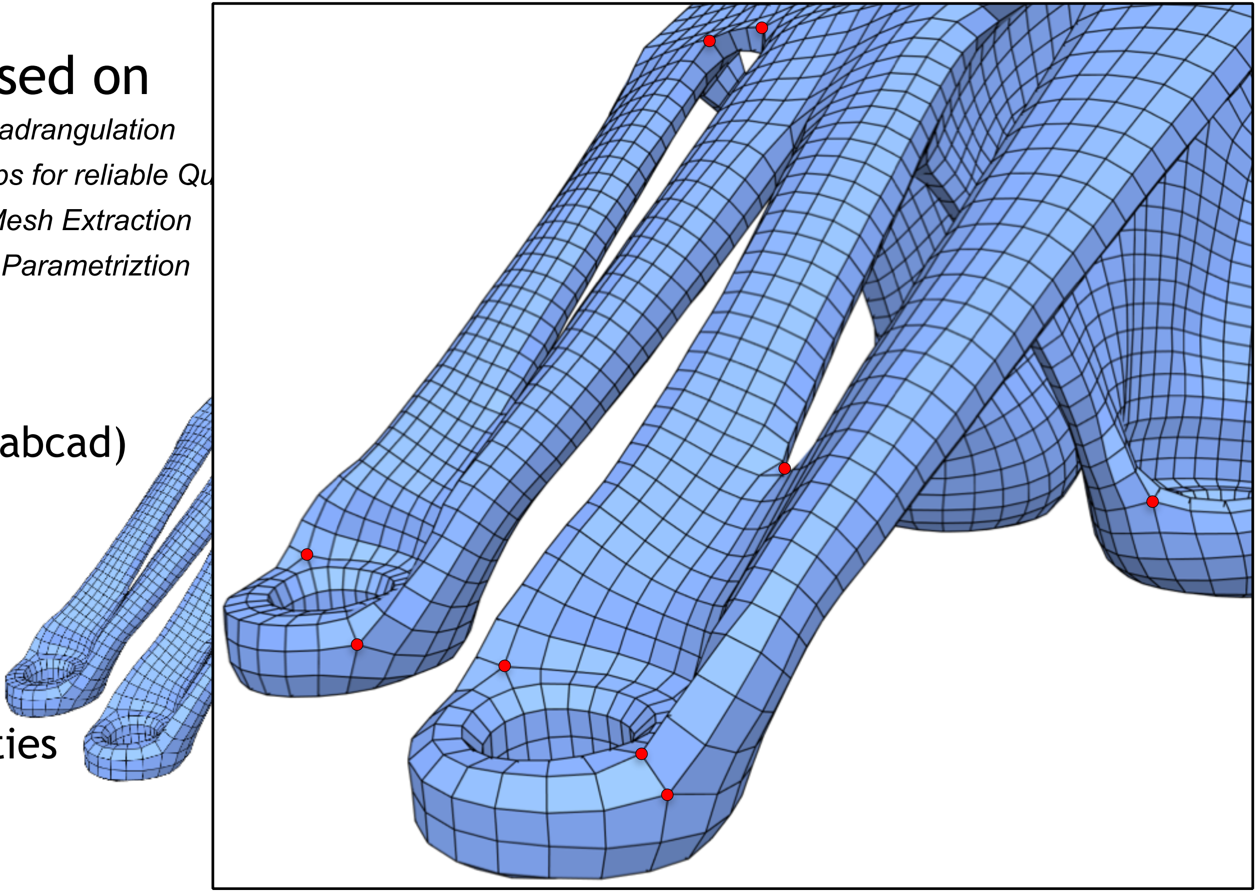
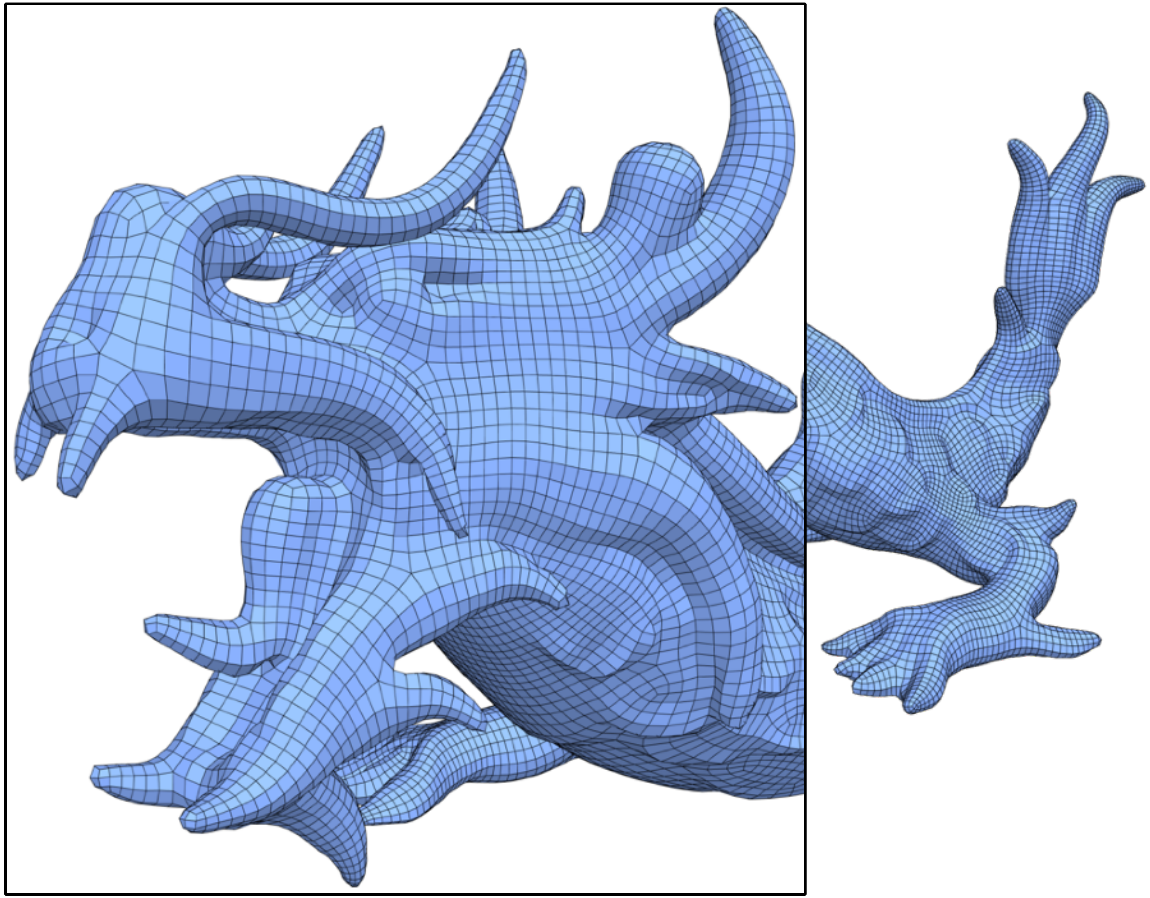


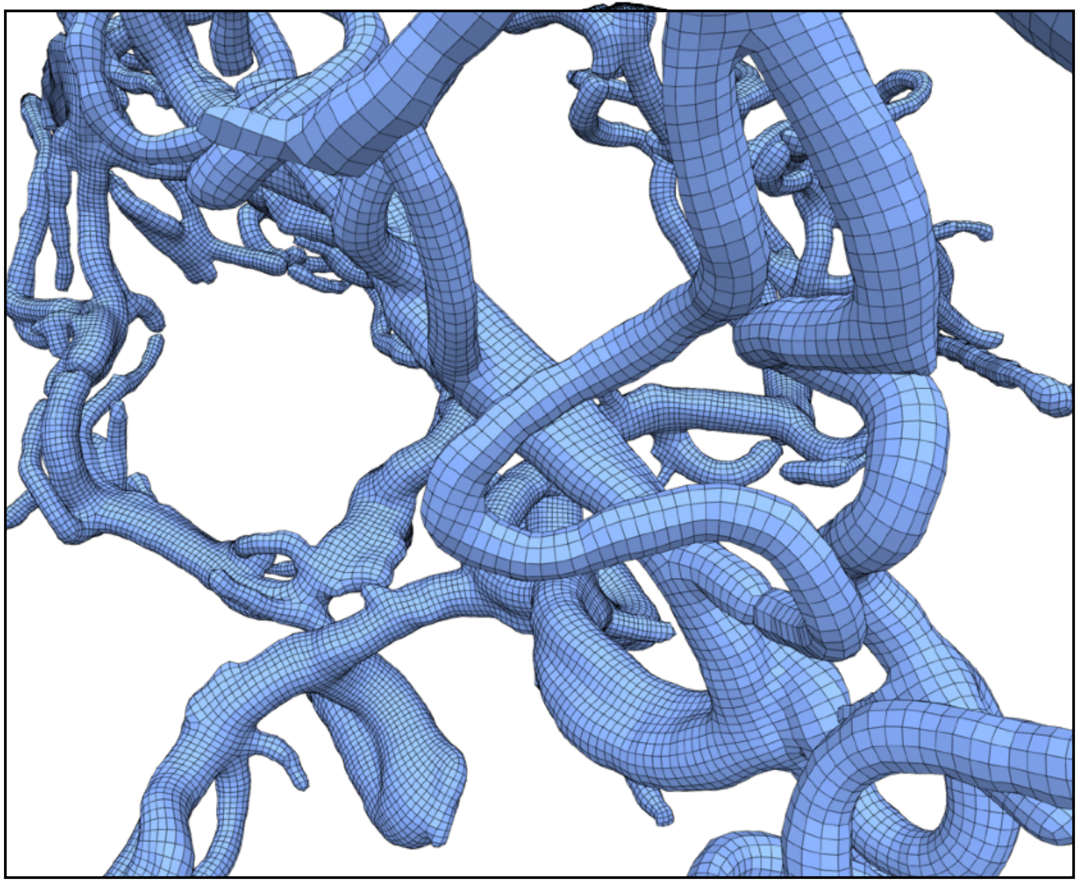
# Results — one-click solution

- asian dragon
- **input**:
  - #triangles 140k
  - many geometric details
- **output**:
  - #singularities 624
  - #quads 27k
  - runtime **90s**

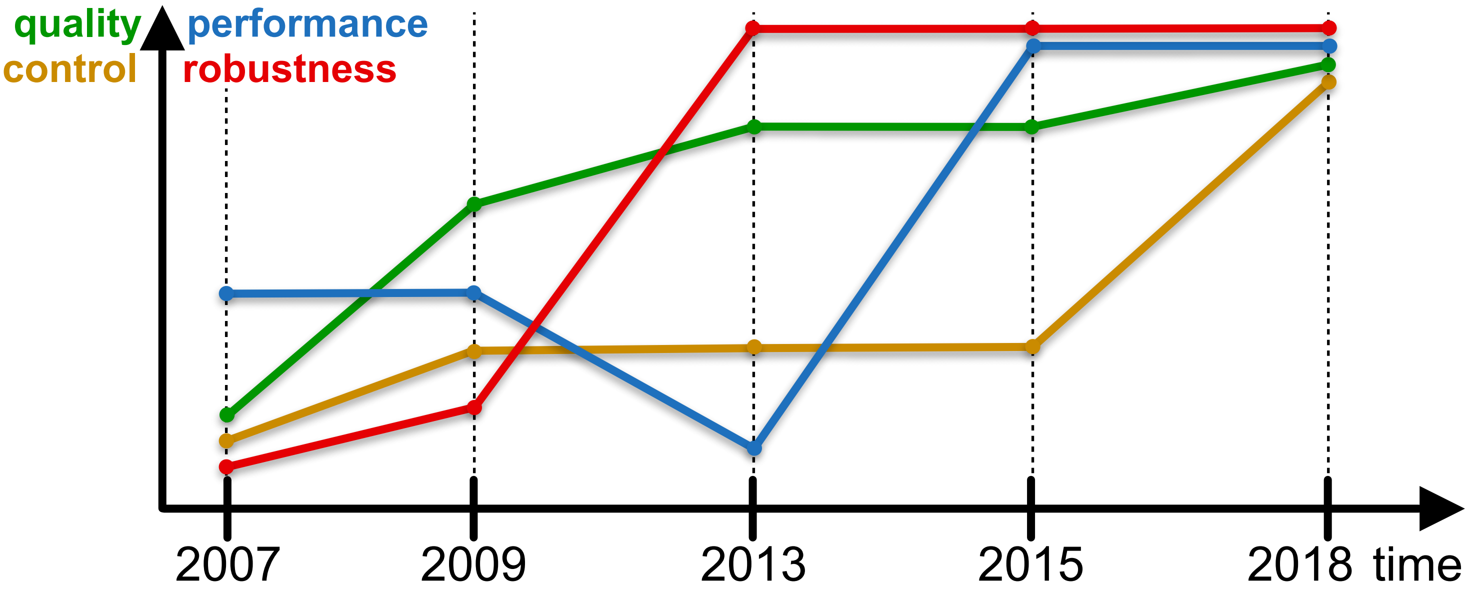


# Results — one-click solution

- vascular structure
- **input**:
  - #triangles 224k
  - tubular network
- **output**:
  - #singularities 1851
  - #quads 125k
  - runtime **3.8min**



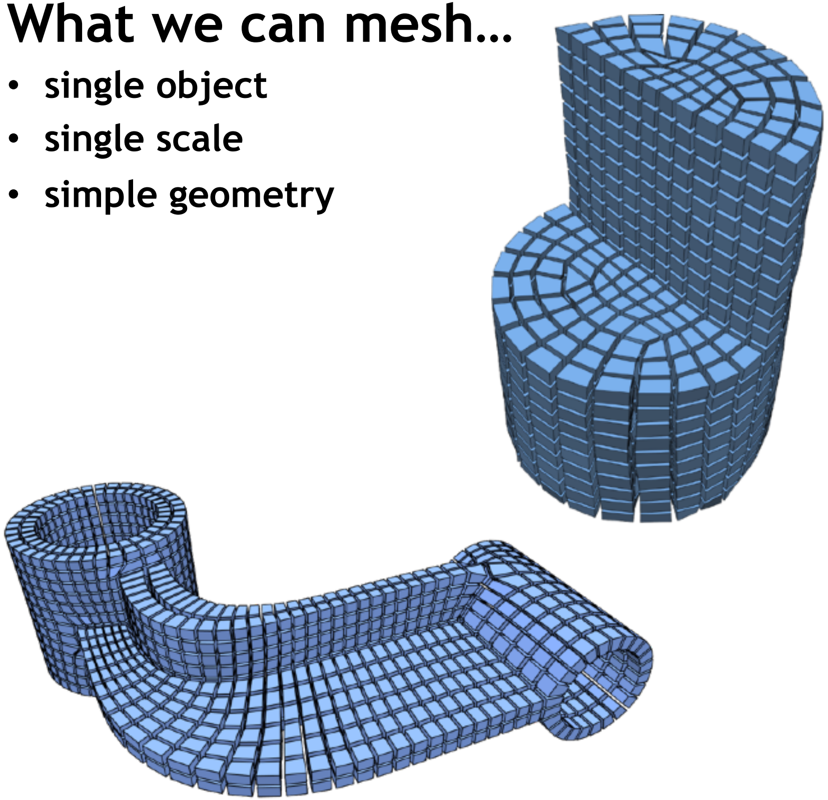
# Evolution of Integer-Grid Map Algorithms



# State of the Art – Hexahedral Meshing with IGMs

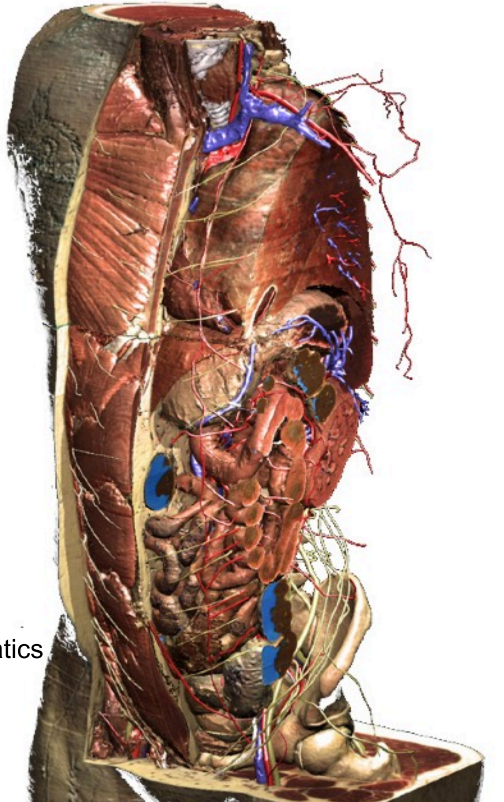
## What we can mesh…

- single object
- single scale
- simple geometry



## What we would like to mesh …

- nested objects
- multiple scales
- complex geometry



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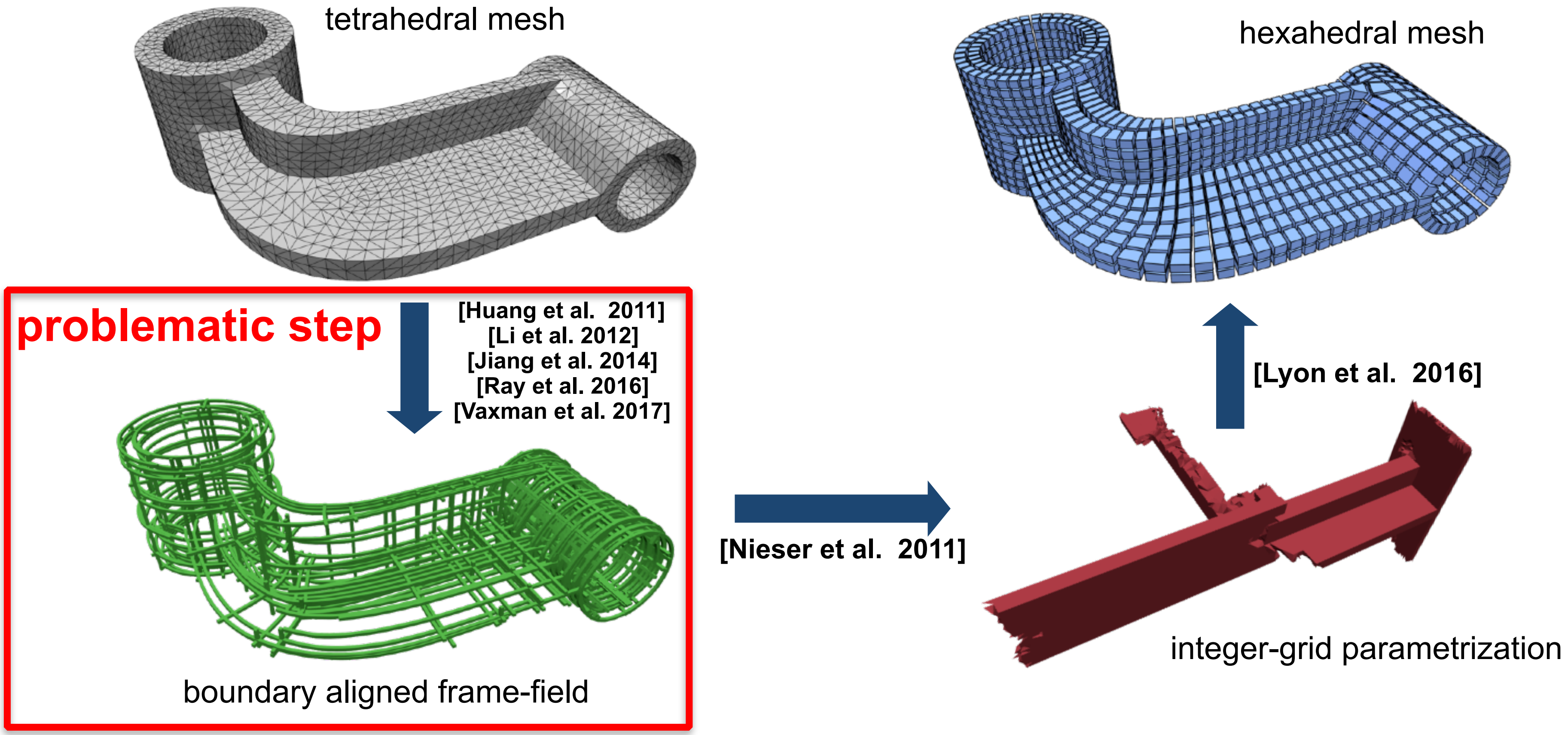
# Hexahedral Meshing via Integer-Grid Maps

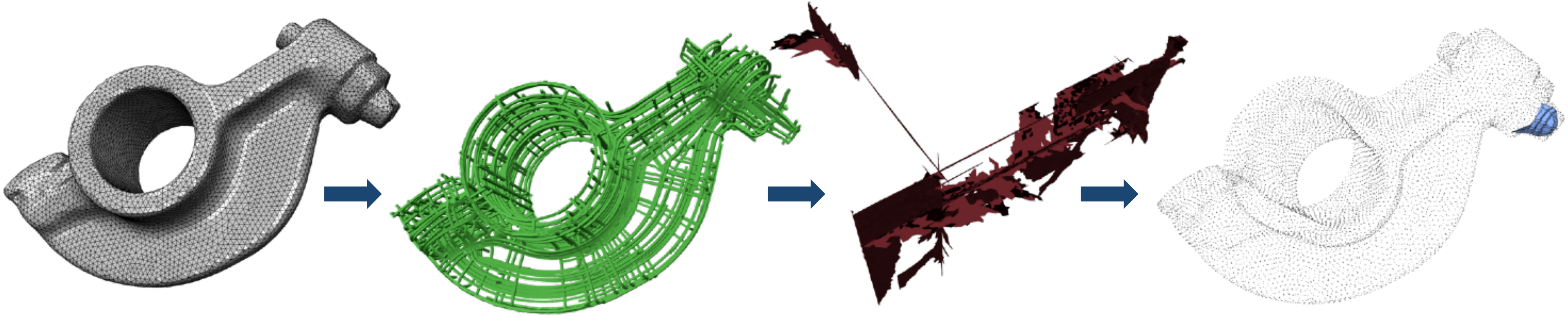
# Hexahedral Meshing via IGM



# Fundamental Topological Problem

**hexahedral mesh singularities** $\subset$ **frame-field singularities**

**invalid singularities ➡ integer-grid map degenerates**



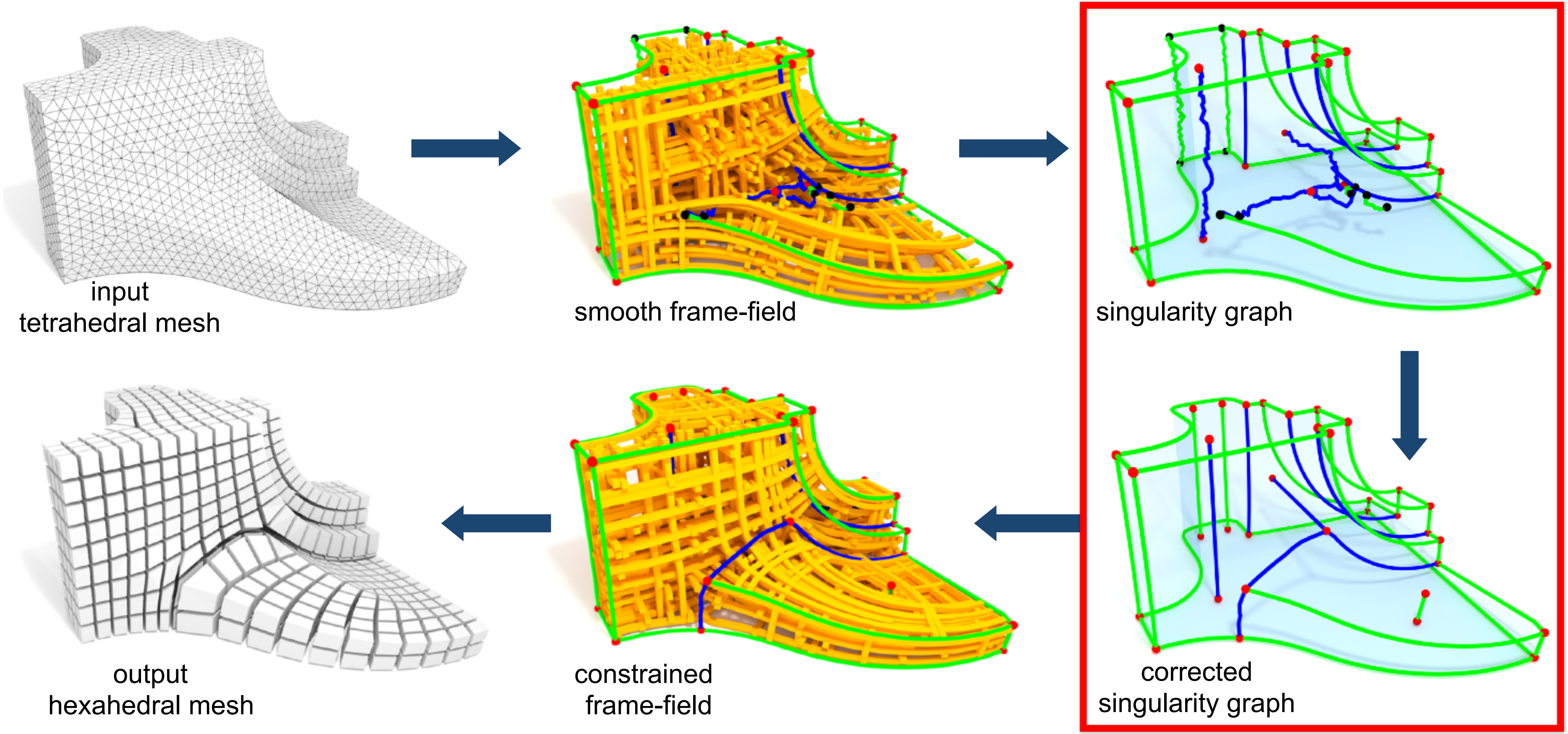
# Singularity-Constrained Octahedral Fields for Hexahedral Meshing

[Liu, Zhang, Chien, Solomon, Bommes — ACM Siggraph 2018]

RWTH AACHEN UNIVERSITY
Visual Computing Institute

Massachusetts Institute of Technology

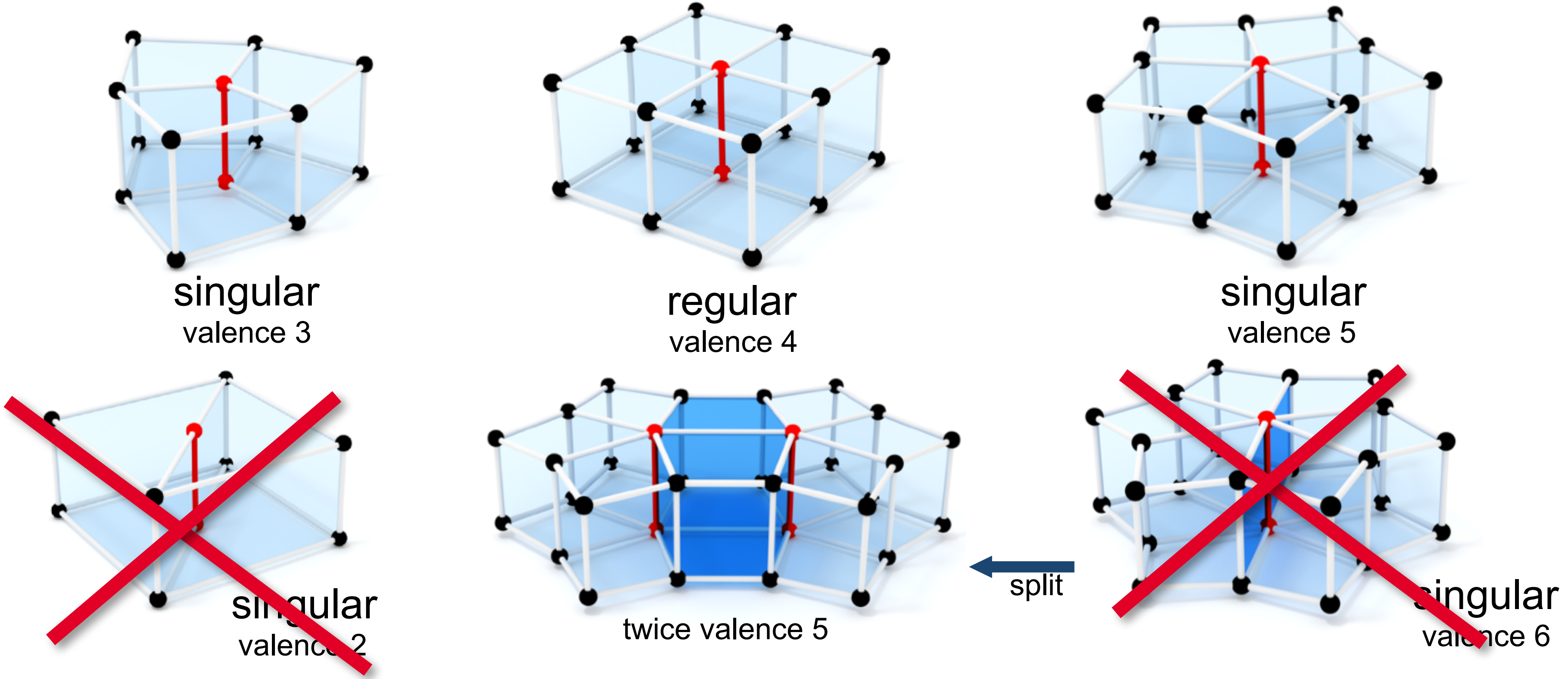
# Modified Algorithm



# Hex-Meshable Singularity Graphs?

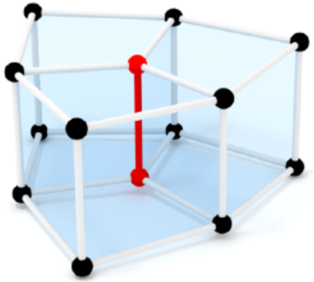
# Hexahedral Mesh Singularities

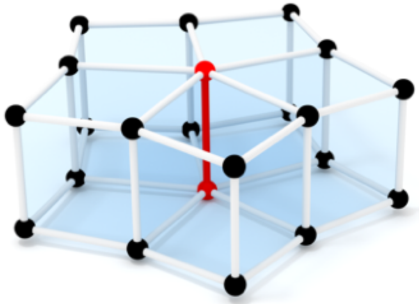
- Hexahedral mesh edges

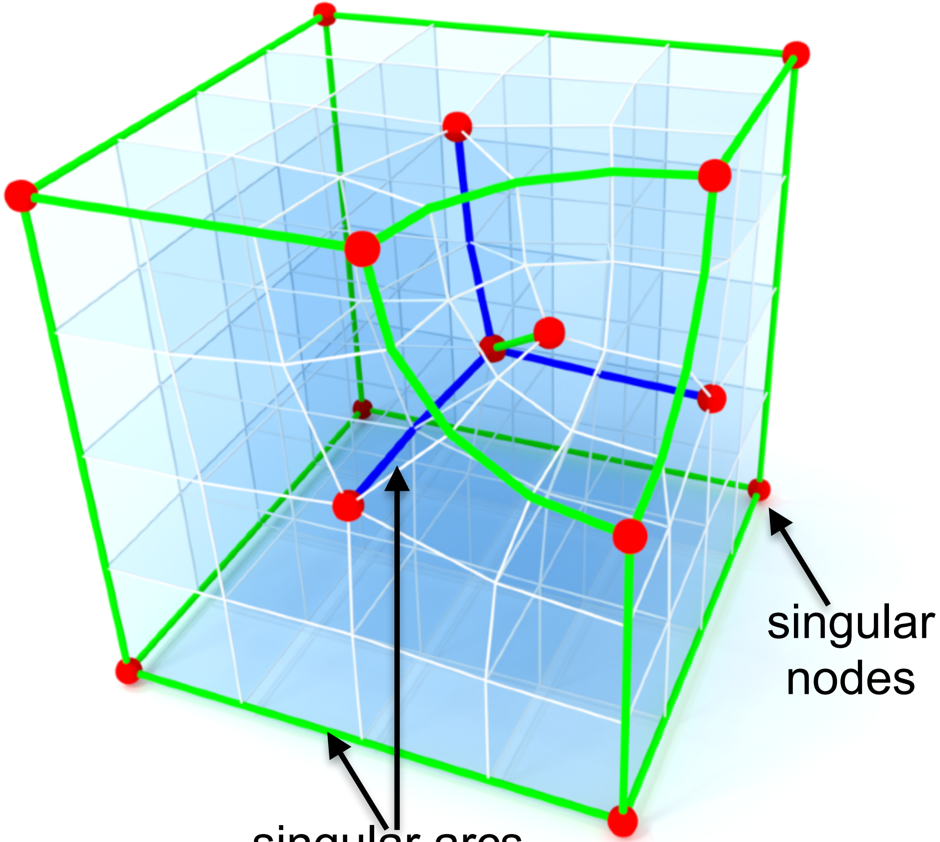


# Hexahedral Mesh Singularities

- Singularity graph



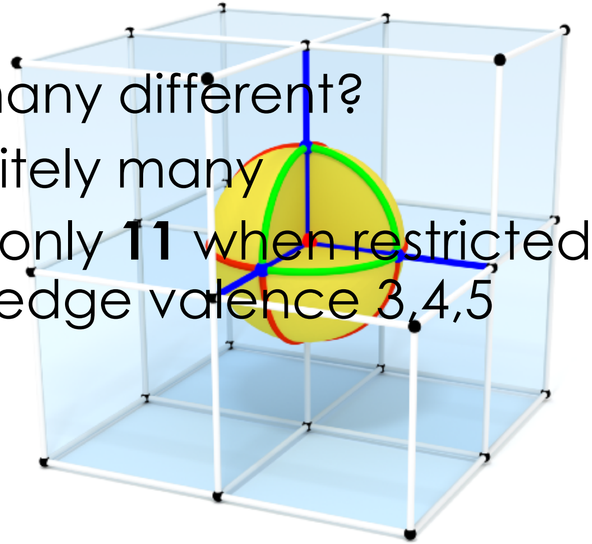


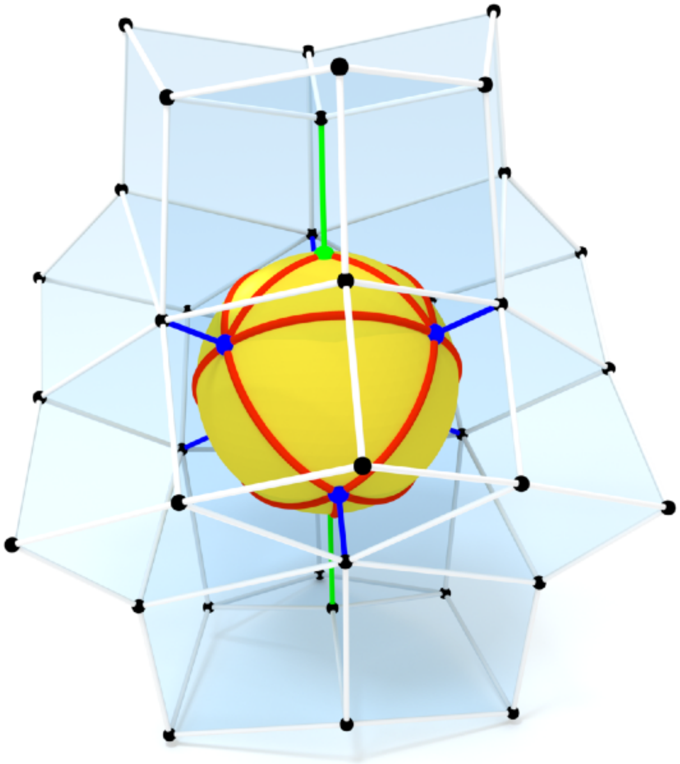


# Hexahedral Mesh Singularities

- **Observation [Nieser et al. 2011]:** hex mesh vertices are isomorphic to triangulations of the sphere

- How many different?
  ➡ infinitely many
  ➡ but only **11** when restricted to hex-edge valence 3,4,5





# Hexahedral Mesh Singularities

- How many sphere triangulations exists with vertex valences restricted to 3, 4, 5?
- Answer: **only 11**
- Assume triangulation has #V vertices with signature (i, j, k)

#valence 3 → i

#valence 4 → j

#valence 5 → k

Euler formula
#V - #E + #F = 2

↓

3i + 2j + k = 12
with
i+j+k = #V

<u>consequences:</u>

1. minimal #V = 4 (i=4)
2. maximal #V = 12 (k=12)

# Hexahedral Mesh Singularities

$3i + 2j + k = 12$
with
$i+j+k = \#V$

- **#V=4**
  (4,0,0)
- **#V=5**
  ~~(3,1,1)~~, (2,3,0)
- **#V=6**
  ~~(3,0,3)~~, (2,2,2), ~~(1,4,1)~~, (0,6,0)
- **#V=7**
  (0,5,2), (1,3,3), ~~(2,1,4)~~
- **#V=8**
  (0,4,4), ~~(1,2,5)~~, (2,0,6)
- **#V=9**
  (0,3,6), ~~(1,1,7)~~
- **#V=10**
  (0,2,8), ~~(1,0,9)~~
- **#V=11**
  ~~(0,1,10)~~
- **#V=12**
  (0,0,12)

valence 5 with 5 vertices requires self-connection

**[Schmeichel and Hakimi 1977]**
"On Planar Graphical Degree Sequences"

**[Mishra and Sarvate 2007]**
"A note on Non-Regular Planar Graphs"

# Hexahedral Mesh Singularities

- **#V=4**
  **(4,0,0)**
- **#V=5**
  (2,3,0)
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  (2,2,2), (0,6,0)
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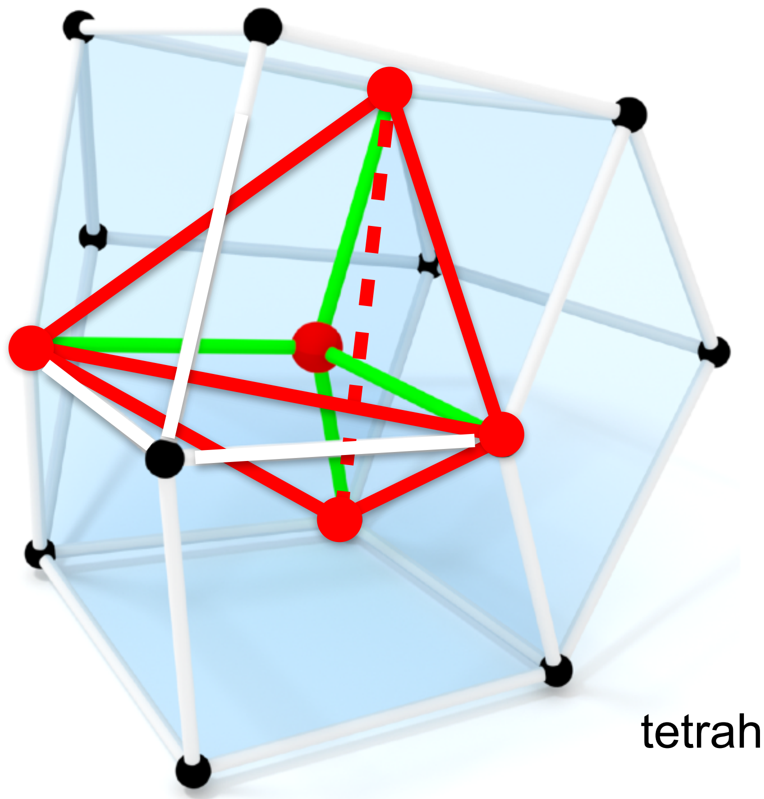

tetrahedron

# Hexahedral Mesh Singularities

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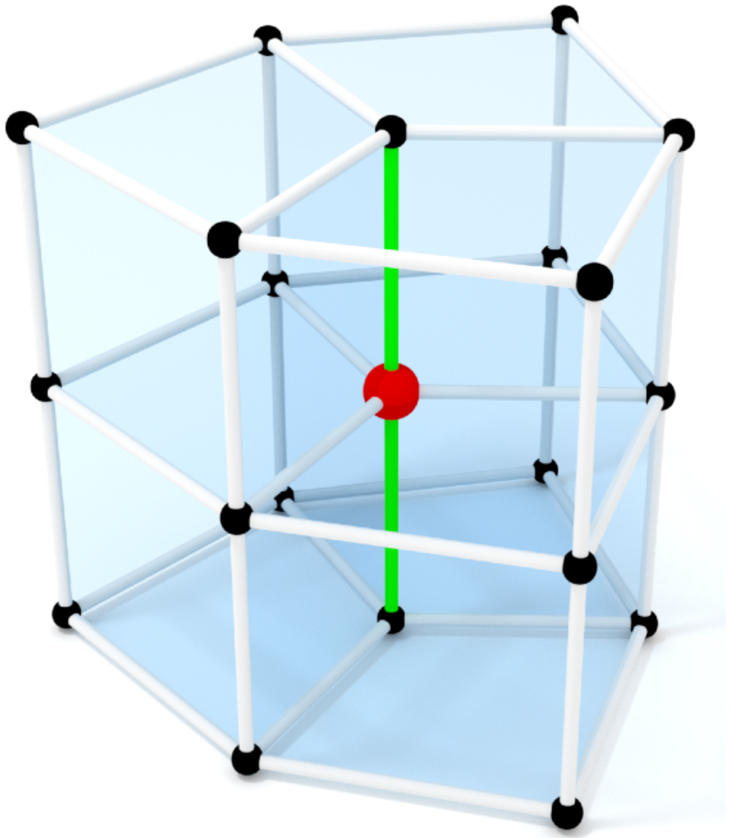

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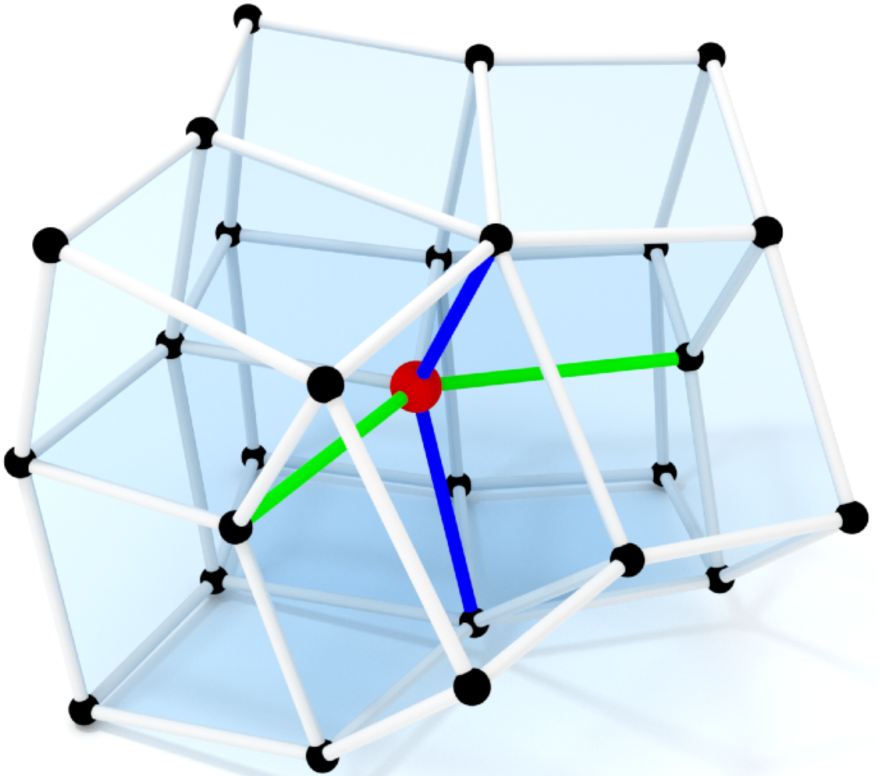

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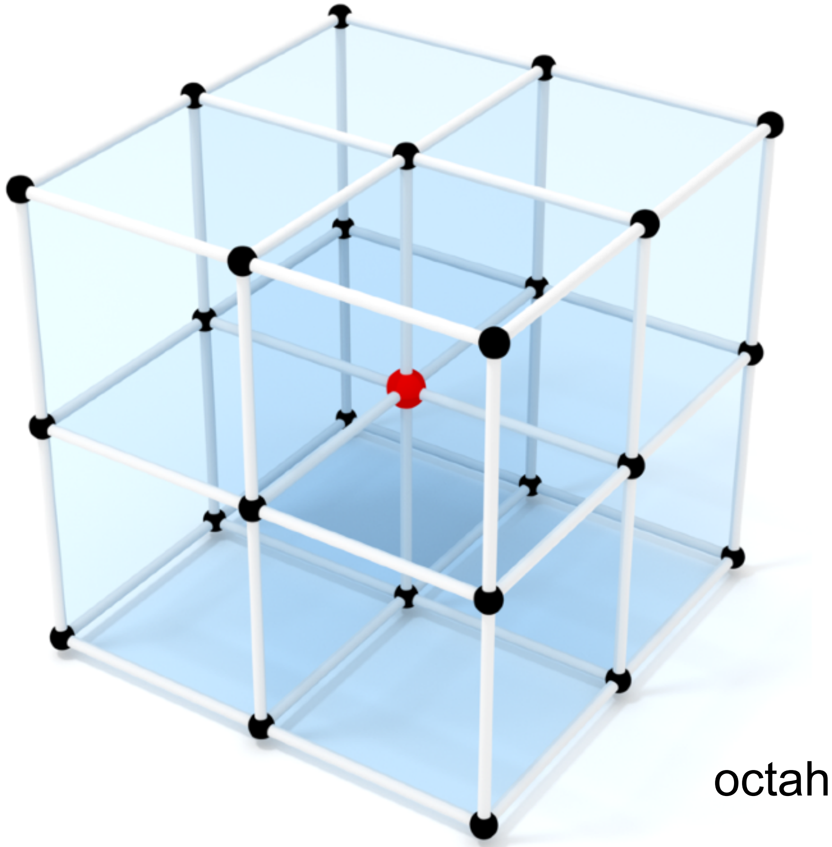

octahedron

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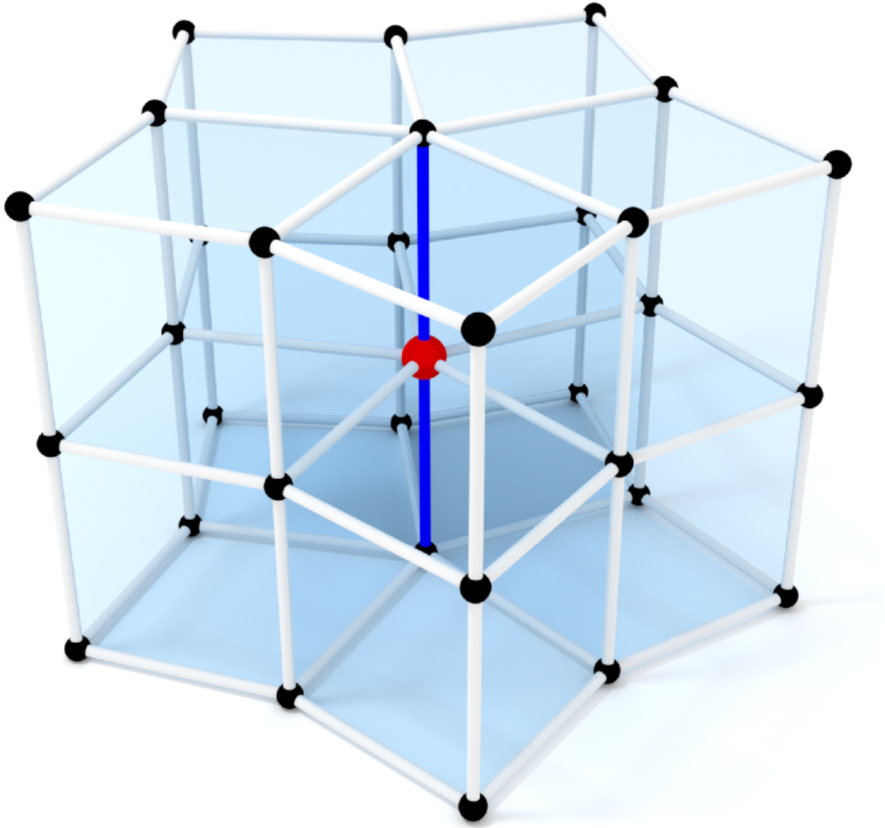

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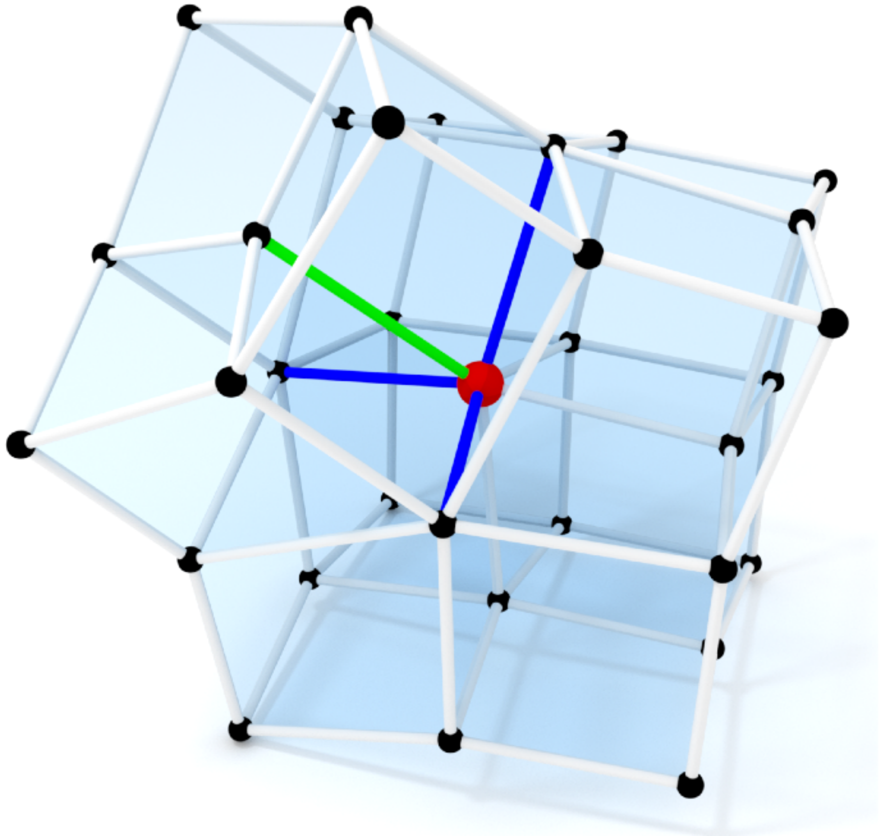

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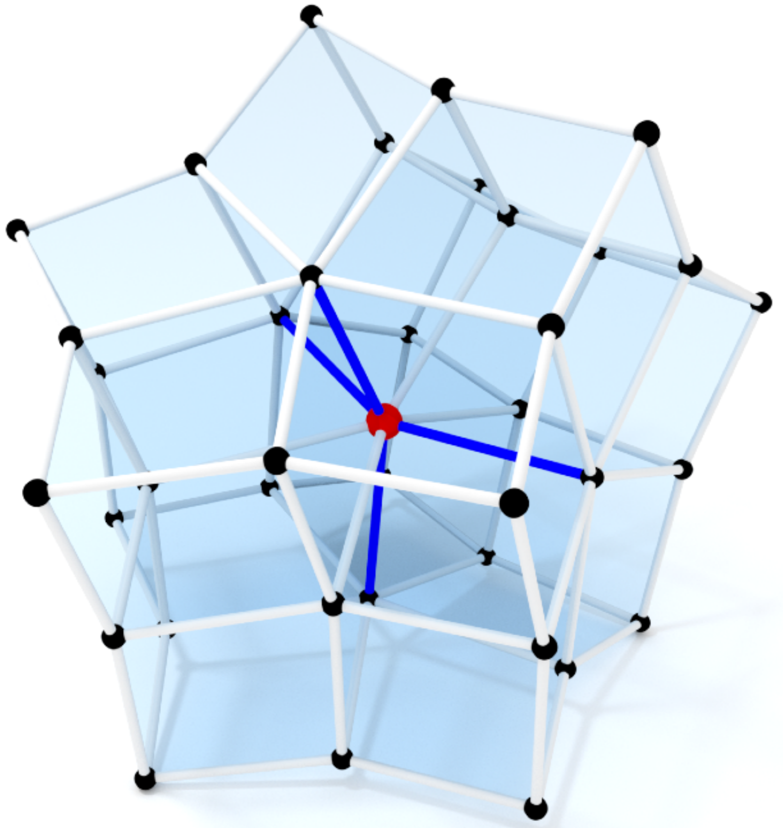

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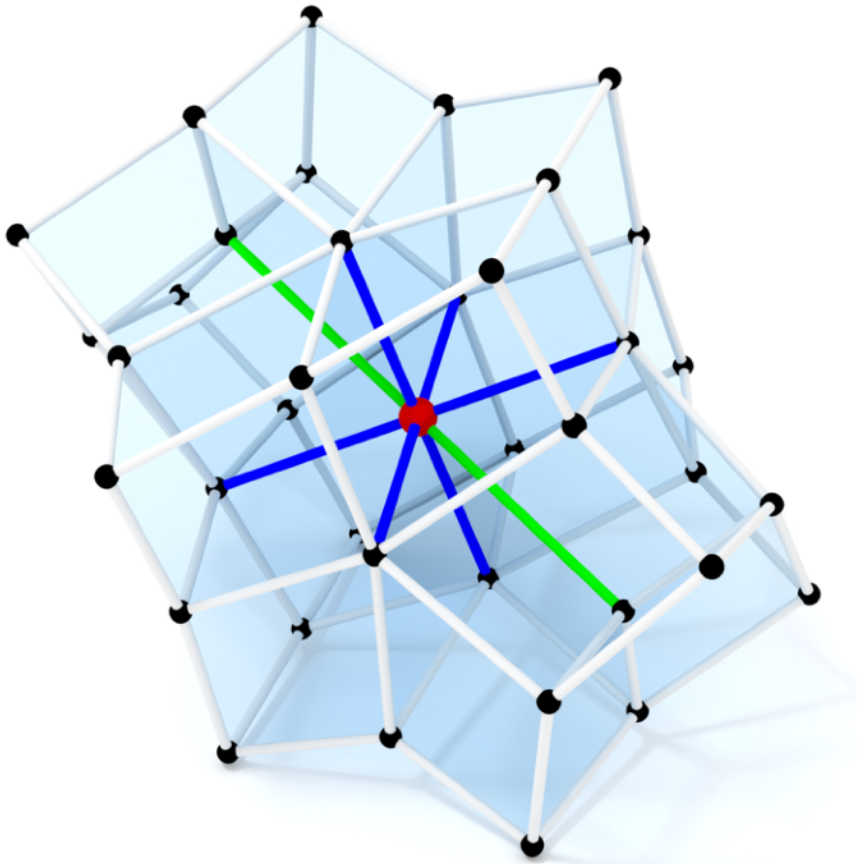

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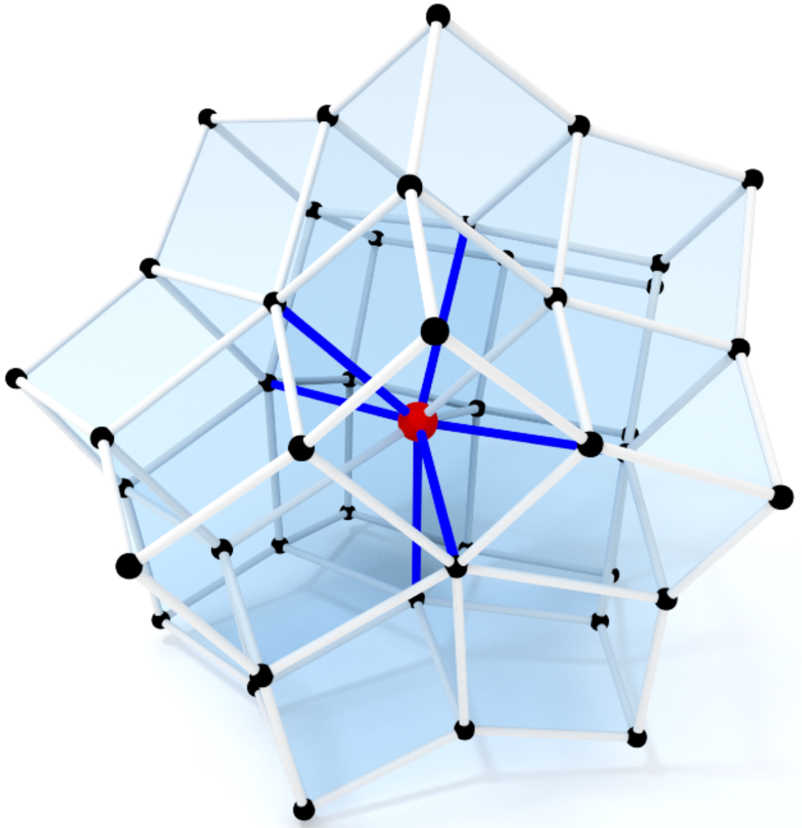

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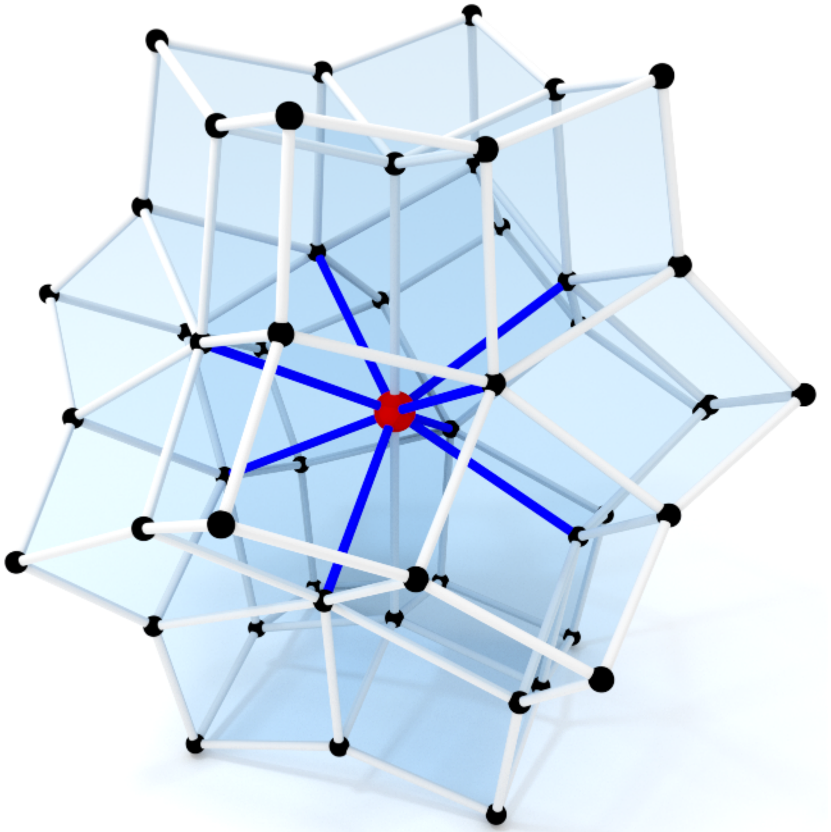

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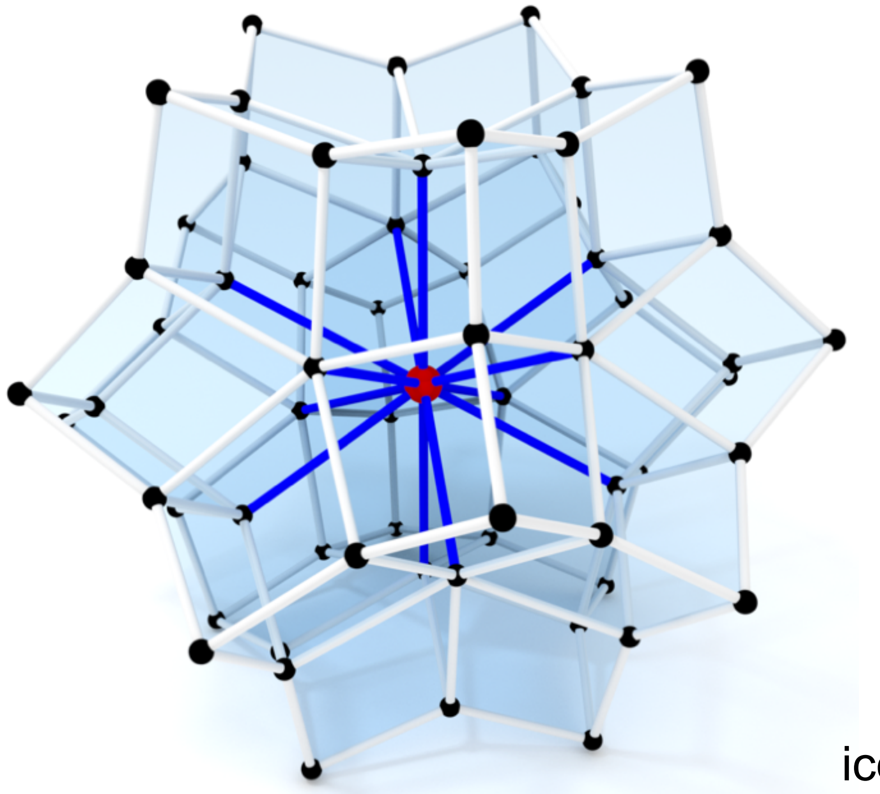
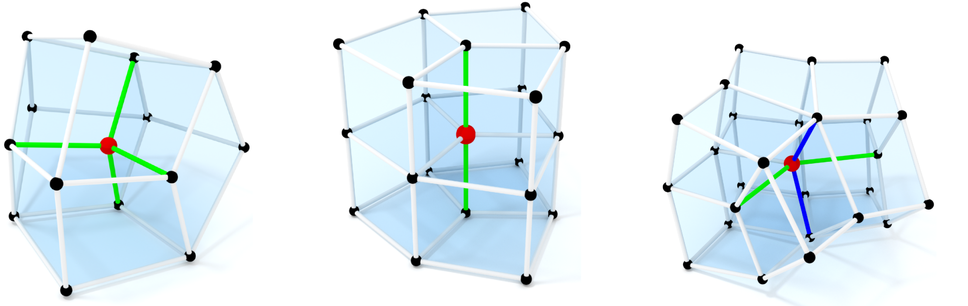
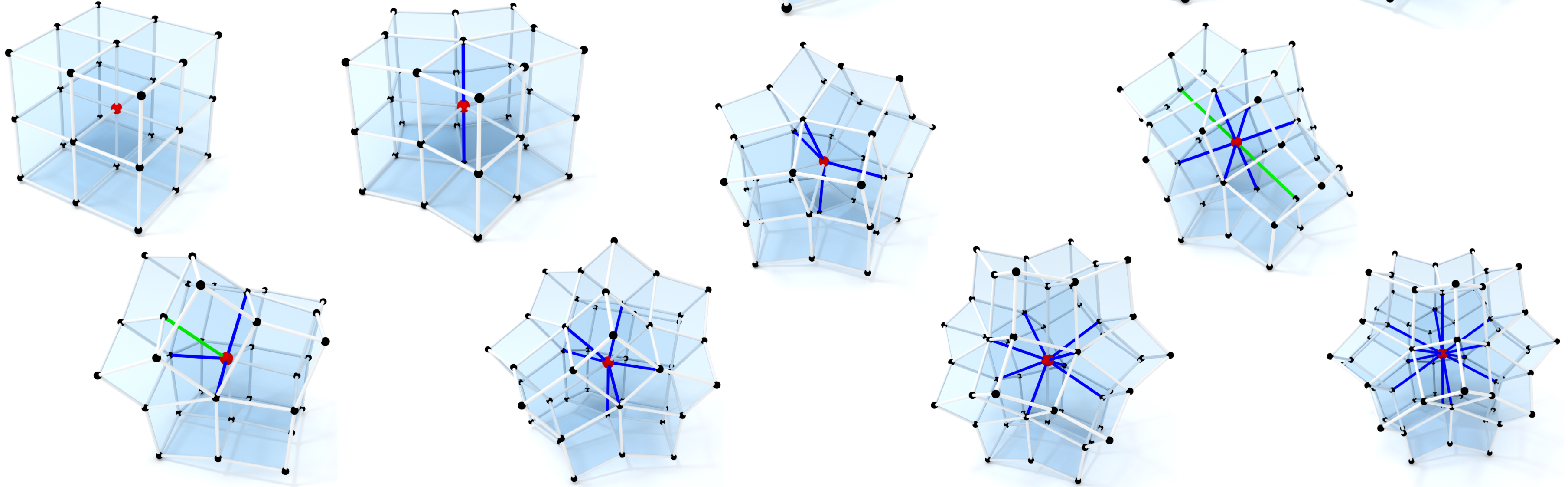


icosahedron

# Hex Meshable Singularity Graphs (valence 3/4/5)

- **Local Necessary Condition**
  each singular node is one of the 11 configurations

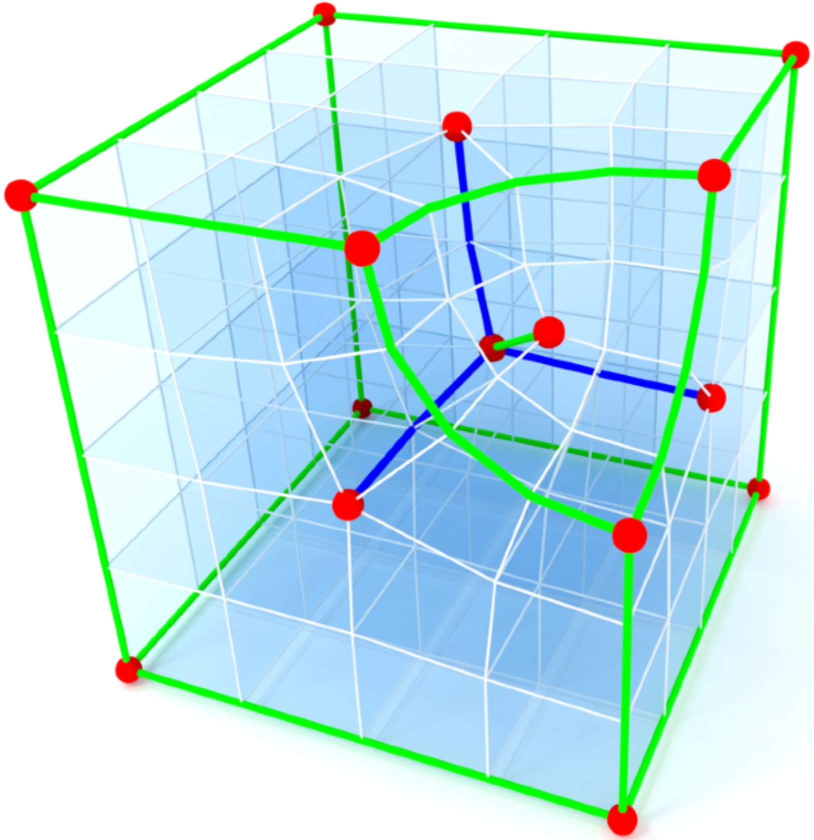




# Hex Meshable Singularity Graphs (valence 3/4/5)

- **Global Necessary Condition**
  analog of discrete Poincaré-Hopf formula

$$\sum_{v \in \partial V_S} \frac{1}{2}\left(1 - \frac{\mathrm{val}_h(v)}{4}\right) - \sum_{e \in \partial E_S^-} \mathrm{idx}(e)$$
$$+ \sum_{v \in \mathring{V}_S} \left(1 - \frac{\mathrm{val}_h(v)}{8}\right) - \sum_{e \in \mathring{E}_S^-} \mathrm{idx}(e) = 0.$$

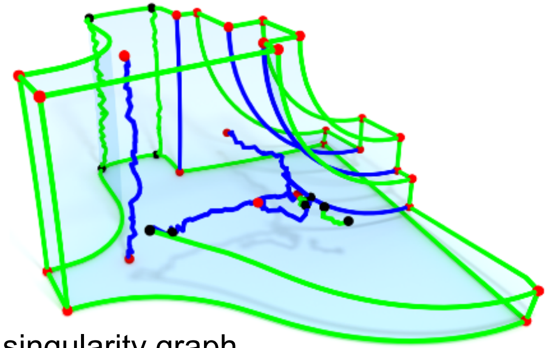


# Hex Meshable Singularity Graphs (valence 3/4/5)

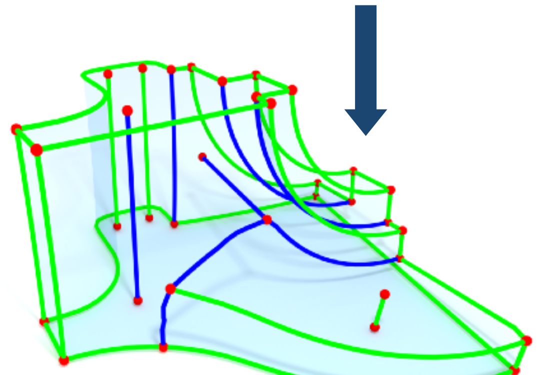
- **Necessary Conditions**
  many inconsistencies can be identified

- **Repairing Singularity Graph**
  open problem
  ➡ so far manually using above conditions

- **Conditions are not sufficient!**
  but algorithmic verification by constructing the corresponding frame-field

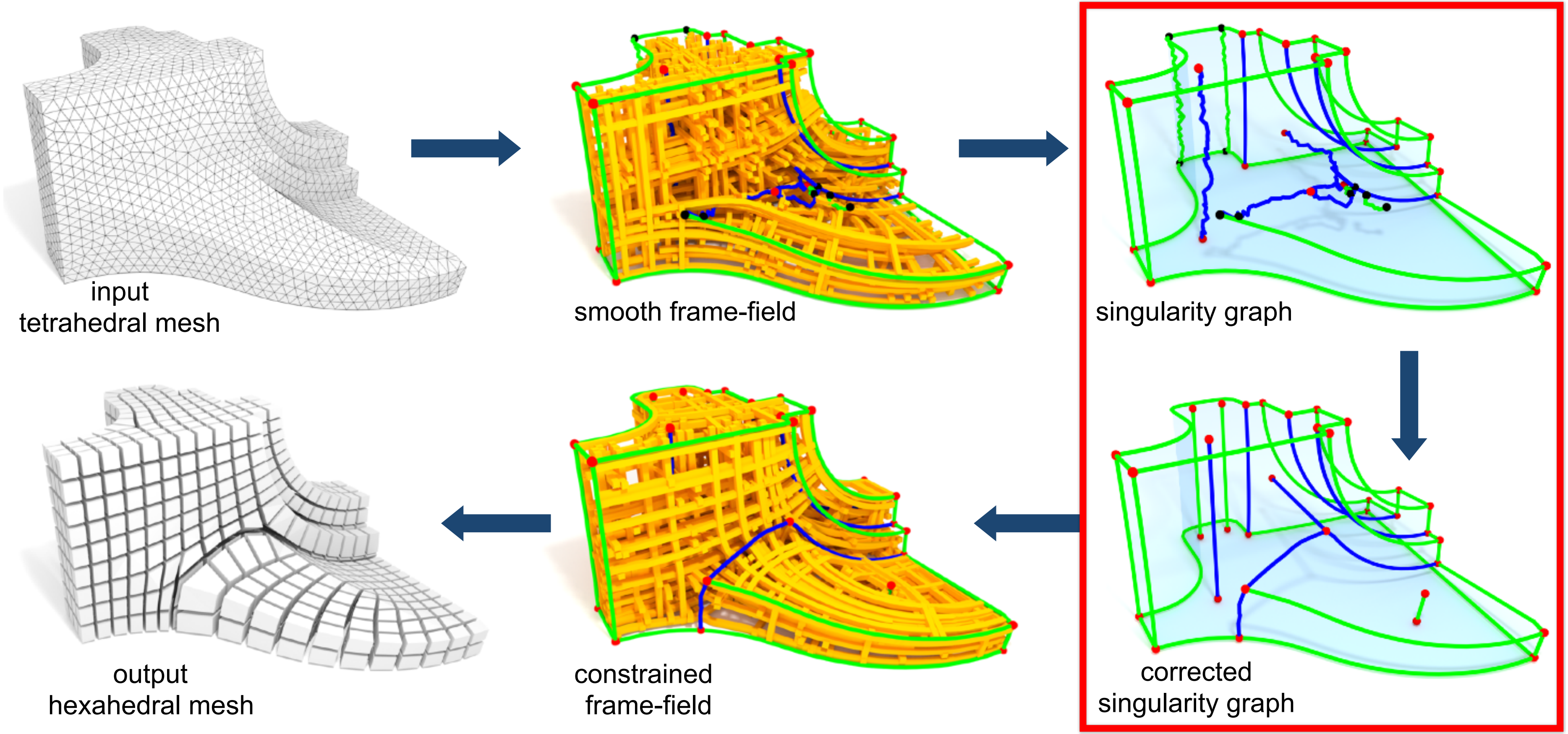


singularity graph

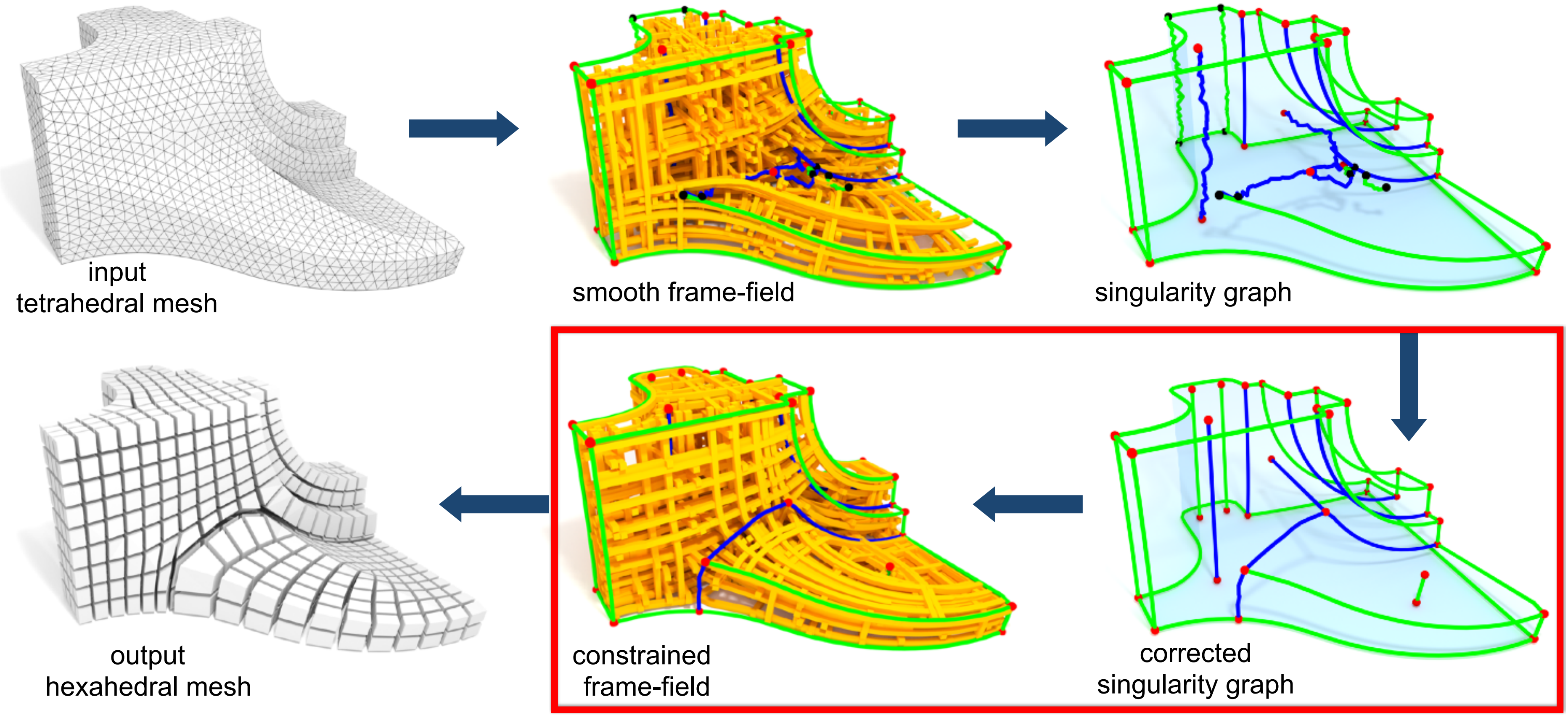


hex-meshable singularity graph

# Modified Algorithm



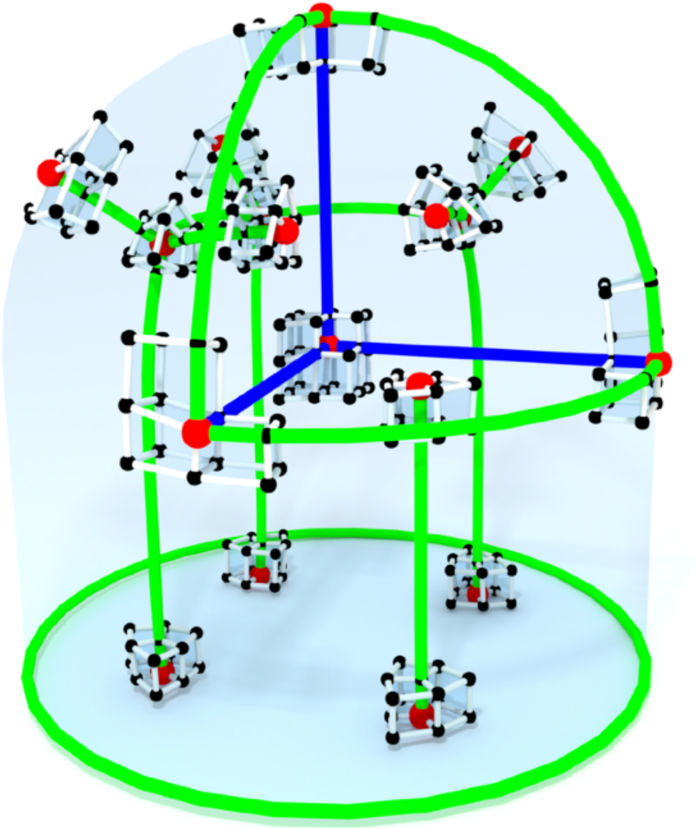
# Modified Algorithm



# Algorithm: Constrained Frame-Fields

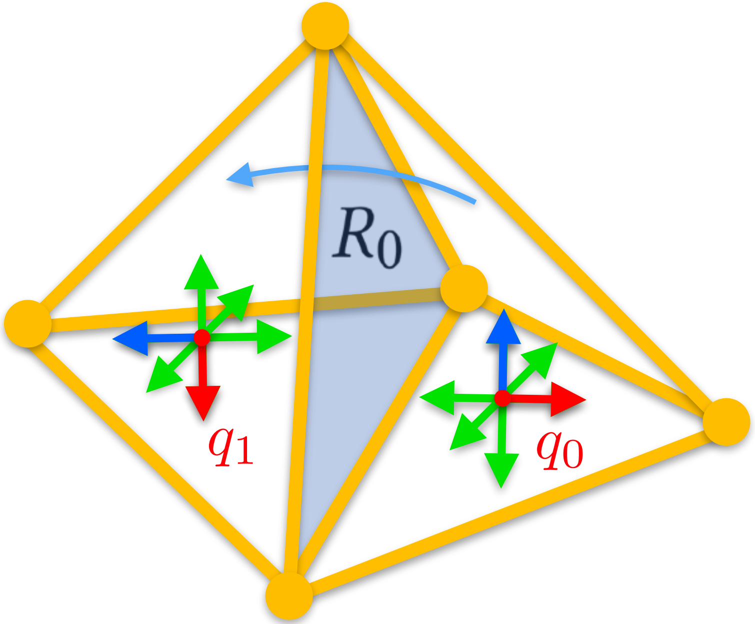
- **Input**:
  - Tetrahedral mesh
  - Singularity graph $\mathcal{S} = (V_S, E_S)$ satisfying
  - ➡ local necessary conditions
  - ➡ global necessary condition

- **Output**:
  - the smoothest discrete cross-field C that is boundary-aligned and matches the singularity graph $\mathcal{S}$



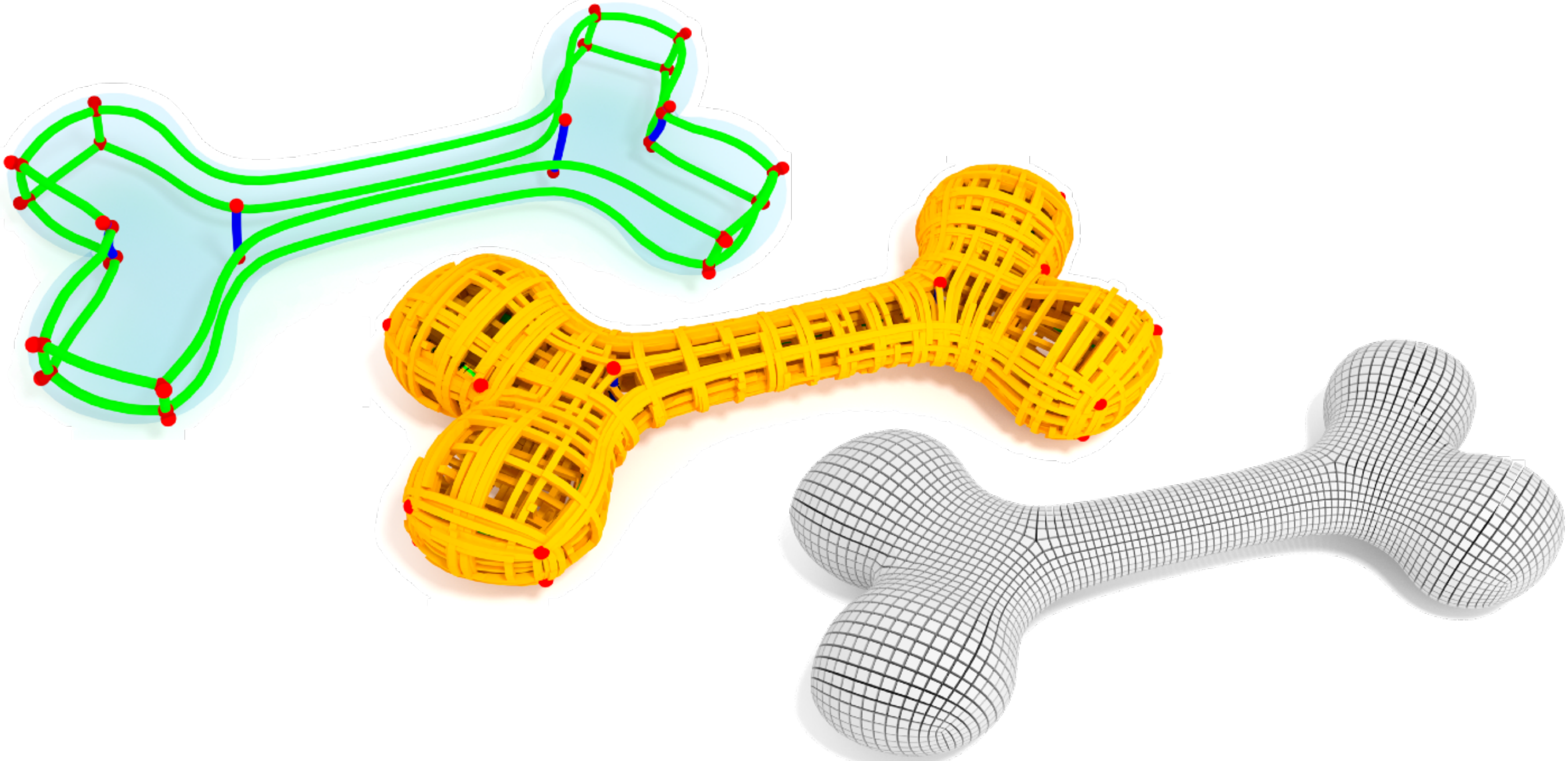
# Algorithm: Constrained Frame-Fields

- **Discrete Frame-Field Representation**
  - one frame $q_i$ per tet
  - one matching $R \in \mathrm{Oct}$ per face
  - 24 different matchings
- **Large Nonlinear Mixed-Integer Problem**
  - singularities induce constraints on products of matching rotations
  - solution of **discrete** matchings can be done independently of **continuous** frame DOFs
  - fast solution possible through careful strategy
  - ➡ chart-merging algorithm
  - ➡ feasibility certifies **global consistency** of singularity graph

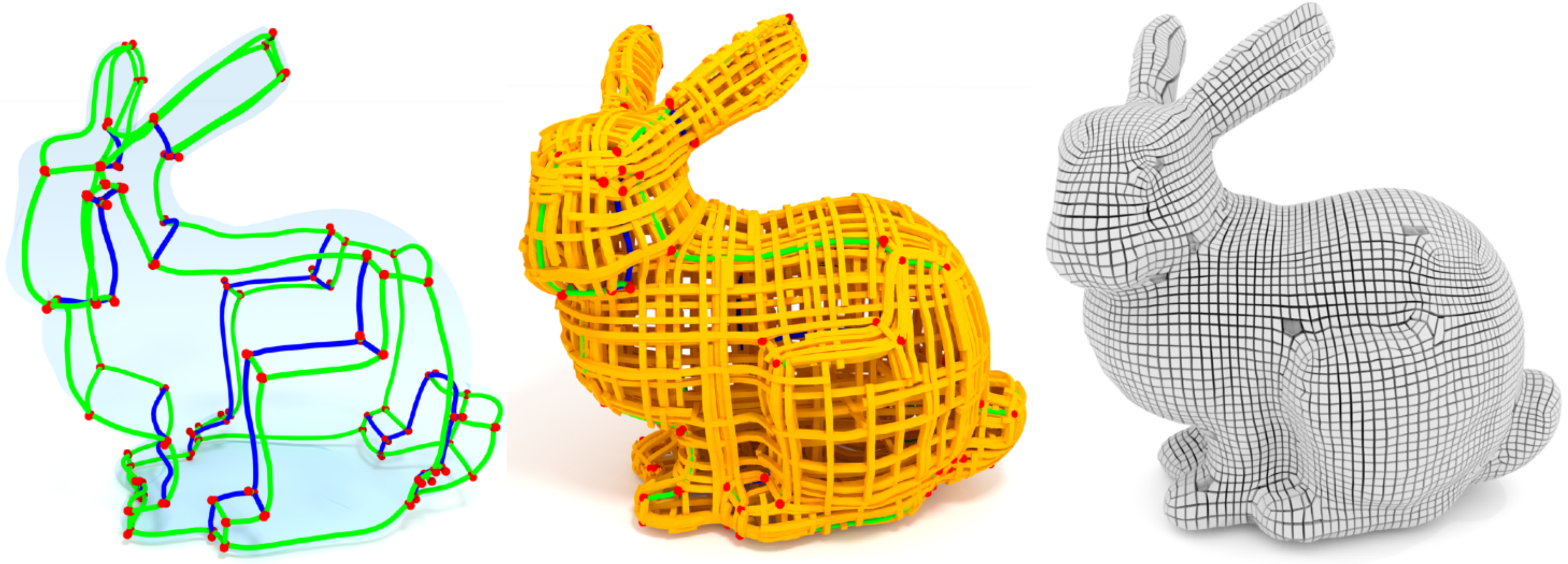


# Results

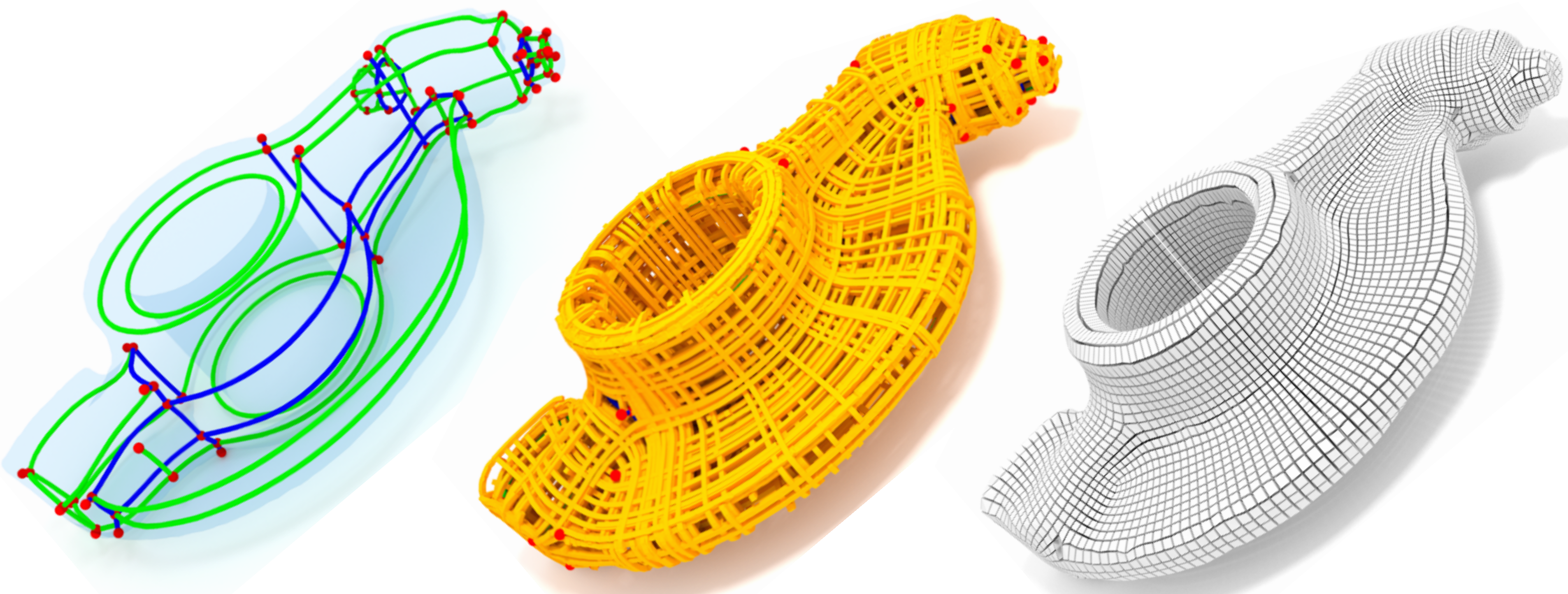
# Results — Bone [71K tets, 0.9s]



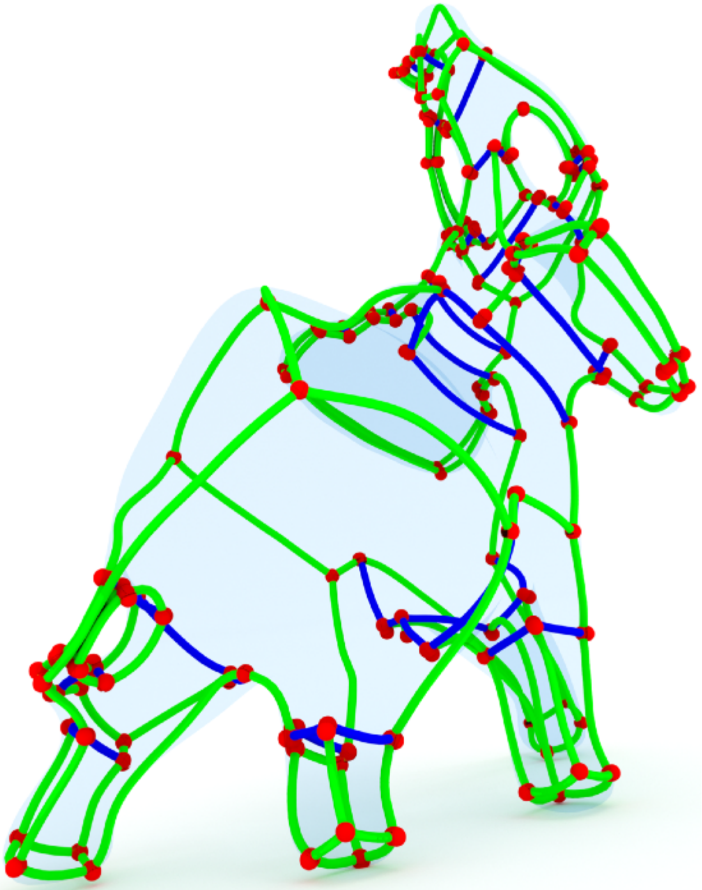
# Results — Bunny [130K tets, 2.7s]



# Results — Rockerarm [122K tets, 3.1s]



# Results — Elephant [300K tets, 9.9s]



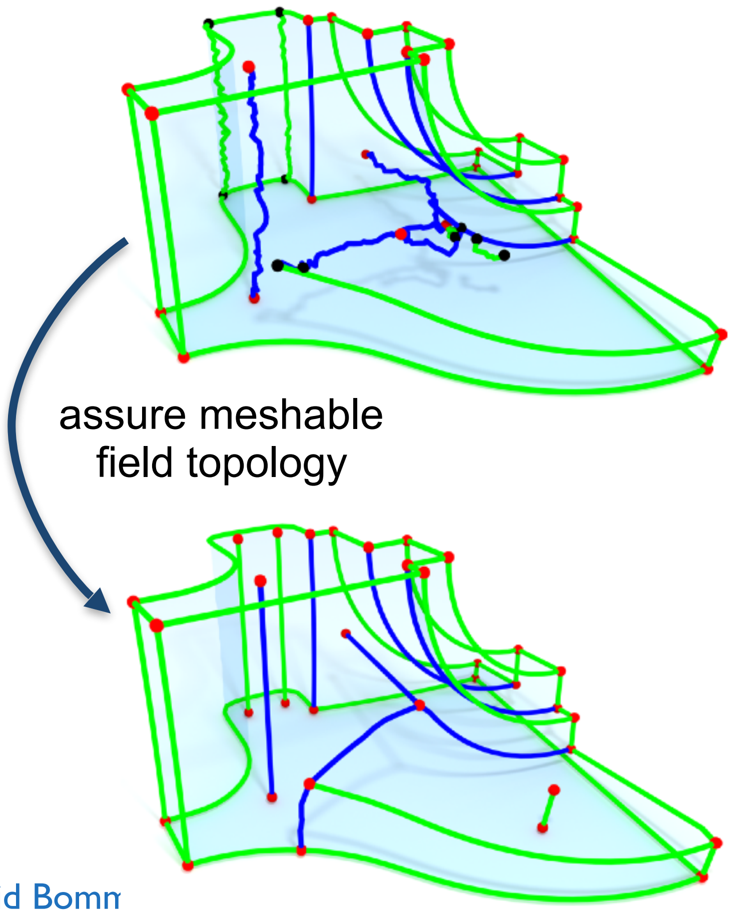
# Integer-Grid Maps
# Scientific Challenges

# Scientific Challenges

**SC1.** Frame-Field Topology
**SC2.** Volumetric Integer-Grid Maps
**SC3.** Precise Control
**SC4.** Quality
**SC5.** Scalability

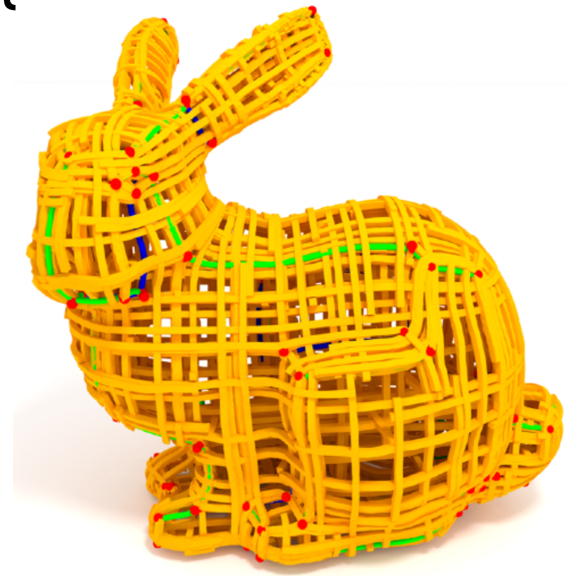
# Scientific Challenges

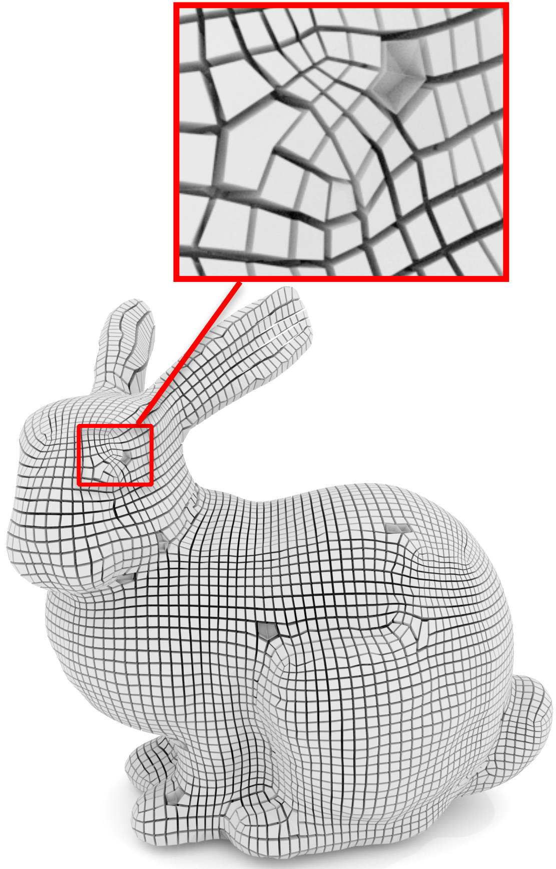
**SC1. Frame-Field Topology**
SC2. Volumetric Integer-Grid Maps
SC3. Precise Control
SC4. Quality
SC5. Scalability



# Scientific Challenges

SC1. Frame-Field Topology
SC2. **Volumetric Integer-Grid Maps**
SC3. Precise Control
SC4. Quality
SC5. Scalability





# Scientific Challenges

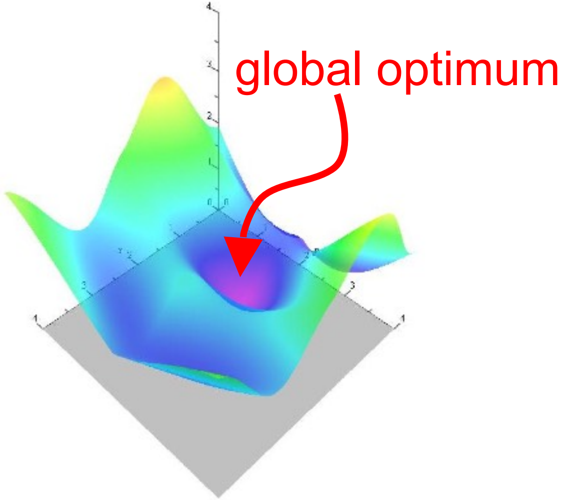
SC1. Frame-Field Topology

SC2. Volumetric Integer-Grid Maps

SC3. **Precise Control** ⟹ **soft & hard constraints**

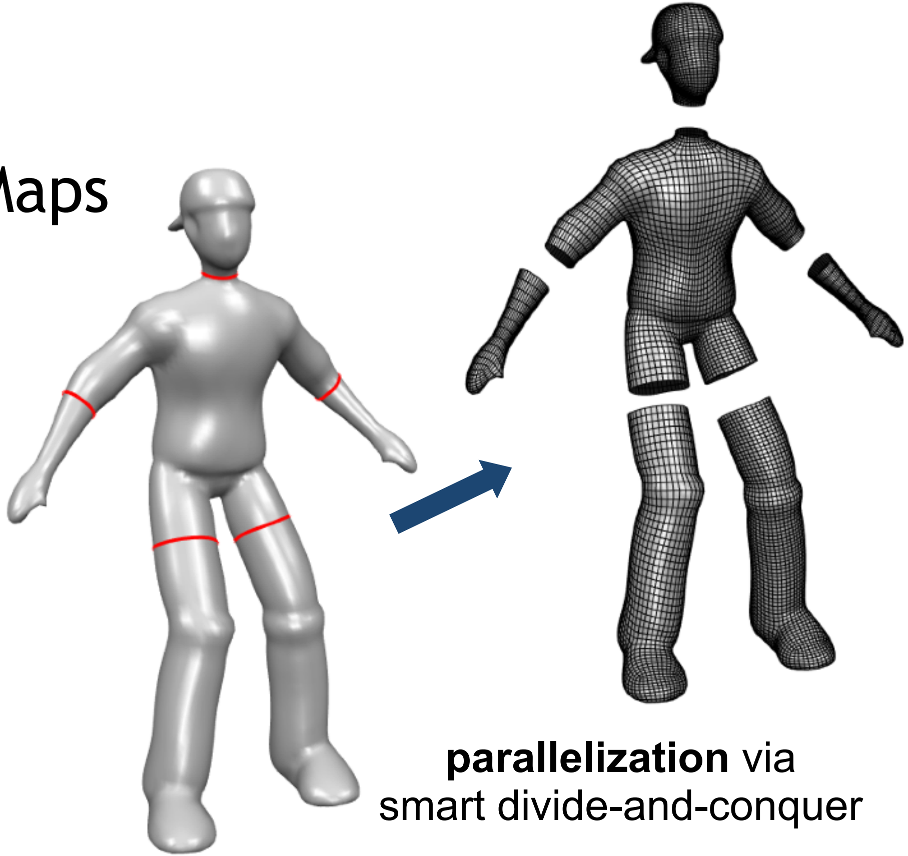
SC4. **Quality** ⟹ **get close to global optimum**

SC5. Scalability



# Scientific Challenges

- SC1. Frame-Field Topology
- SC2. Volumetric Integer-Grid Maps
- SC3. Precise Control
- SC4. Quality
- SC5. **Scalability**



**parallelization** via
smart divide-and-conquer

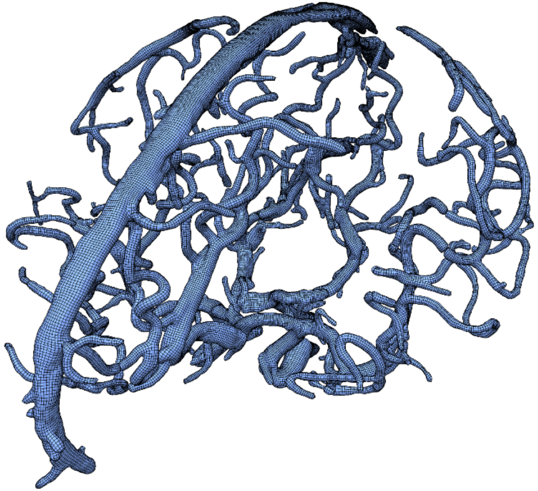
# Summary & Outlook

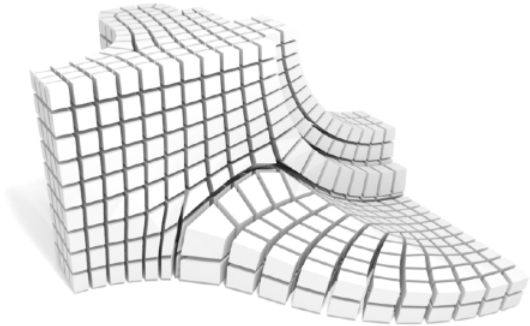
- **Integer-Grid Maps**
  - offer good balance of quality criteria
  - efficiency through frame-fields

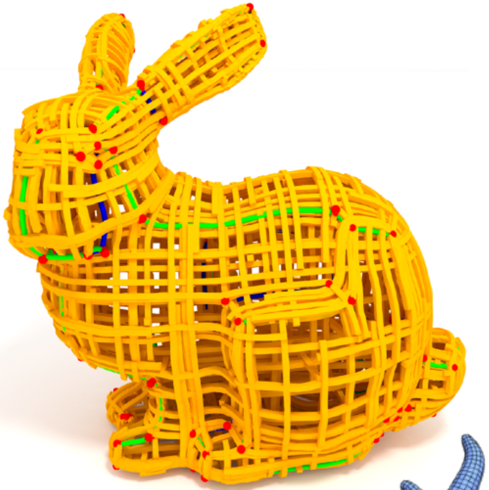
- **Quadrilateral Meshing**
  - robustness / performance / quality / control
  - complete toolbox available

- **Hexahedral Meshing**
  - significantly more challenging, many unsolved aspects, e.g. robustness
  - **contribution 1**: local/global necessary conditions for meshable singularity graphs
  - **contribution 2**: algorithm for singularity-constrained fields
  - important next step towards robustness:

    ➡ automatic correction of singularity graph







# Thank You!