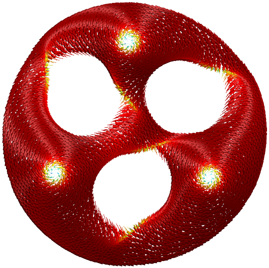
26th International Meshing Roundtable

Computing cross fields
A PDE approach based on Ginzburg-Landau theory

Pierre-Alexandre Beaufort, Jonathan Lambrechts, François Henrotte, Christophe Geuzaine, Jean-François Remacle

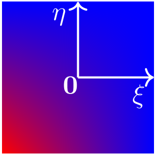


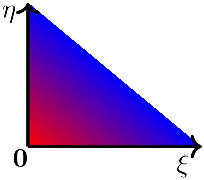
Université catholique de Louvain - iMMC
Université de Liège - Montefiore Institute

Motivation

Meshing quadrangles

Meshing quadrangles for finite elements methods





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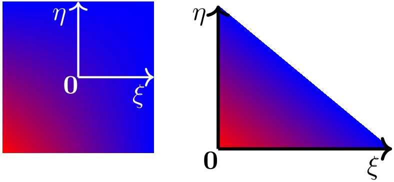
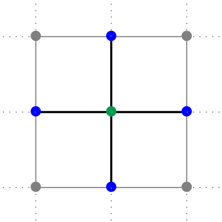


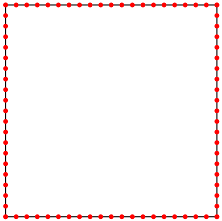
Quadrangle quality strongly depends on point locations



One way to spawn points

Frontal approach

From boundaries,...



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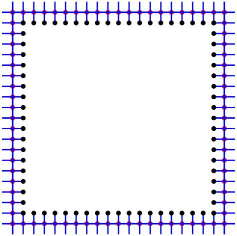

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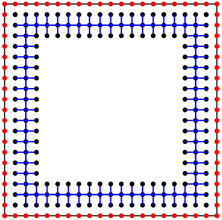

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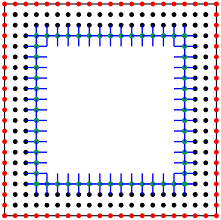
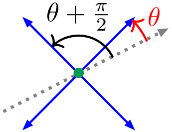


For all $\mathbf{x} \in M$, 4 preferred orthonormal directions are given



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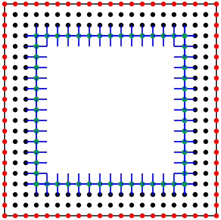

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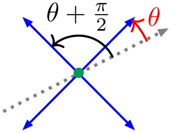
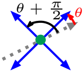


$\implies$ it defines a **cross field**

$$\theta \overset{?}{=} \theta + \frac{\pi}{2}k, \forall k \in \mathbb{Z}$$

Modeling
Cross fields



A cross field which relative angle is θ may be defined by

$$\langle \cos(4\theta); \sin(4\theta) \rangle$$

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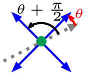

A cross field which relative angle is θ may be defined by

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which is suitable:

- **Uniqueness**
$$\cos\left(4\left[\theta + k\frac{\pi}{2}\right]\right) = \cos(4\ \theta),\ \forall k \in \mathcal{Z}$$
- **Distance**
$$\int_0^{2\pi} |\cos(4[\theta_i + \alpha]) - \cos(4[\theta_j + \alpha])|^2 d\alpha$$
$$= \pi\left((\cos(4\theta_i) - \cos(4\theta_j))^2 + (\sin(4\theta_i) - \sin(4\theta_j))^2\right)$$

Complex analogy
Vector fields

Actually, a cross field consists of vector fields:

$$< \underbrace{\cos(4\theta)}_{u}; \underbrace{\sin(4\theta)}_{v} > \equiv \underbrace{\exp(i\ 4\theta)}_{\exp(i\ \theta)^4} = u + i\ v$$

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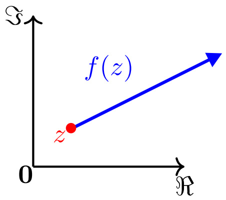
Two dimensional vector fields correspond to values of complex functions



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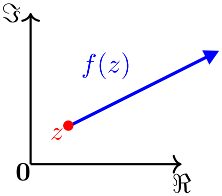
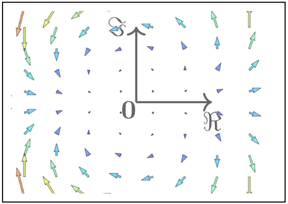
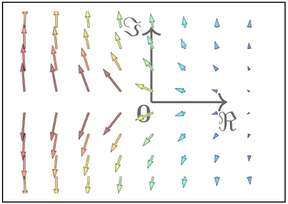


$f(z) = z^2$

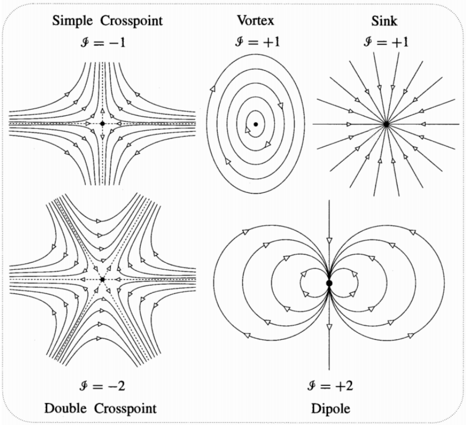




$f(z) = \log(z)$

Critical points
Indices

- Vector fields may have **critical point(s)** z^c, $f(z^c) = 0$
- z^c has an **index** $\mathfrak{I}$ defined as winding number of points around



[1]

[1] Figure from Tristan Needham's book, "Visual complex analysis"

Poincaré-Hopf theorem
Vector fields on closed surfaces

If a vector field on a smooth closed surface of genus g has only a finite number n of singular points s_j, then

$$\sum_{j=1}^{n} \mathfrak{I}(s_j) = 2(1-g)$$

where $2(1-g)$ equals the Euler characteristic of a closed surface

Computing cross fields
Criteria

How is a cross field built over a surface?

- Smooth cross fields for smooth directions
- Average boundary orientations within surface
- *Cross field should have unit norm almost everywhere**

Engineering approach

Energy formulation

Smooth out and average data from boundary conditions with Laplace

$$E(u; v) = \min_{u,v} \int_M |\nabla u|^2 + |\nabla v|^2 d\boldsymbol{x}$$

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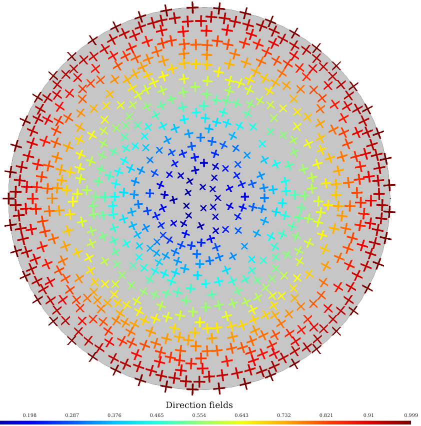
But data vanishes far away boundaries



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Energy formulation

A penalty term is then added to foster unit norm cross fields

$$E(u;v) = \min_{u,v} \underbrace{\int_M |\nabla u|^2 + |\nabla v|^2 d\boldsymbol{x}}_{\text{Smoother term}} + \underbrace{\beta \int_M (u^2 + v^2 - 1)^2 d\boldsymbol{x}}_{\text{Penalty term}}$$

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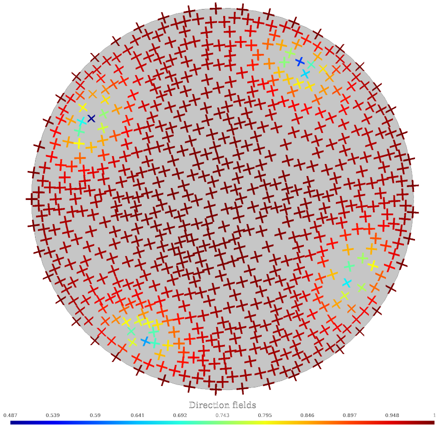

Ginzburg-Landau functional
Preliminaries

Ginzburg-Landau functional is

$$E_\epsilon(f) = \frac{1}{2}\int_M |\nabla f|^2 d\boldsymbol{x} + \frac{1}{4\epsilon^2}\int_M (|f|^2 - 1)^2 d\boldsymbol{x}$$

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$$\min_{f \in H^1_g(M,\mathbb{C})} E_\epsilon(f)$$

Initial mapping
Topological requirements

Let $\mathcal{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$.

$$\min_{f \in H^1_g(M, \mathcal{S}^1)} \int_M |\nabla f|^2 d\boldsymbol{x}$$

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Otherwise, $|\nabla f|^2$ is not bounded in some $\mathbf{x}^c \in M$

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Penalty term

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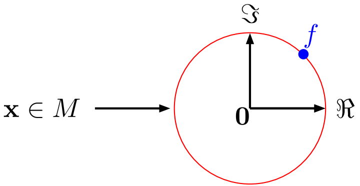
Asymptotic behavior of solution f_ϵ

$$\int_M |\nabla f_\epsilon|^2 d\boldsymbol{x} \xrightarrow[\epsilon \to 0]{} \infty$$

Link with directional fields

Ginzburg-Landau equation

- Ginzburg-Landau functional maps $\mathbf{x} \in M$ over unit circle $\mathcal{S}^1$



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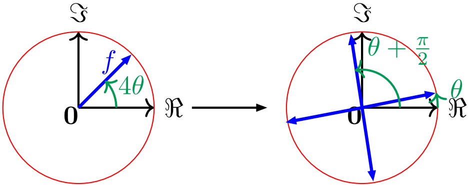

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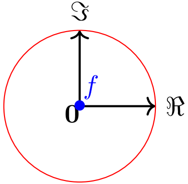
- Asymptotic behavior of Ginzburg-Landau equation yields vector fields critical points



$\Longrightarrow$ undefined directions

Critical points
Ginzburg-Landau functional

A critical point z^c has following contribution

$$\pi \left(\Im(z^c)\right)^2 |\log(\epsilon)|$$

as ϵ tends to zero within Ginzburg-Landau functional E_ϵ

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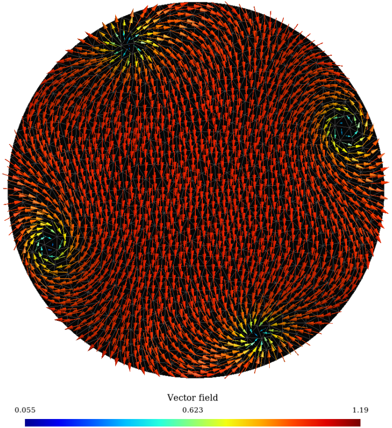

Non unit norm cross field

Weak constraint

Since $|f|$ may be different of unity, it has to be redefined

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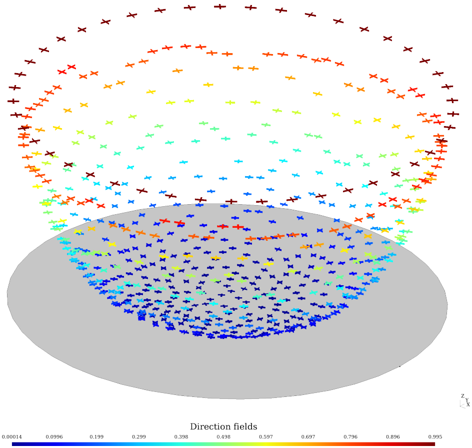

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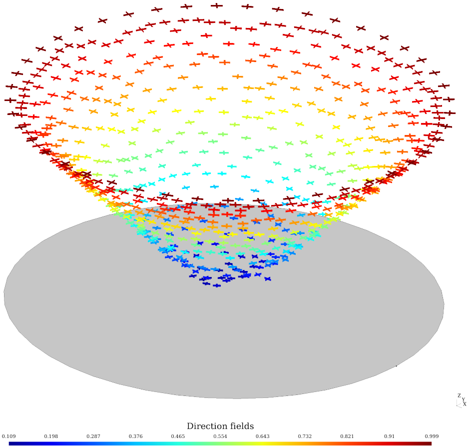

Directional fields
Interpretation

Directions of cross fields correspond to 4-th roots of vector fields expression

$$f(z) = z^4 = r^4 \exp(i\ 4\theta)$$

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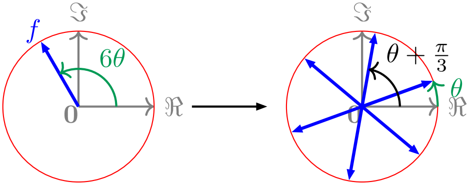
Directions of n-fields correspond to n-th roots of vector fields

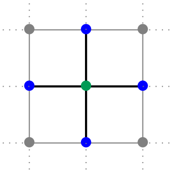
$$f(z) = z^n = r^n \exp(i\ n\theta)$$

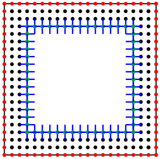
Directional fields with 6 symmetries spawn vertices of equilateral triangles

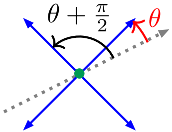


Conclusion

- Quadrangle quality may be ensured by a cross field







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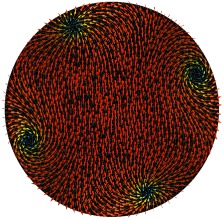

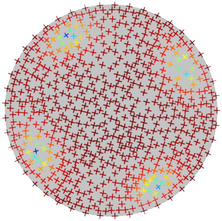

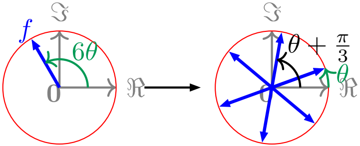

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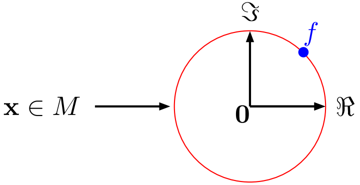

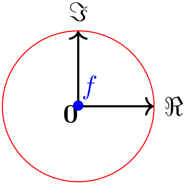

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Conclusion

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- A n-directional field corresponds to n-th root of a vector field
- $H^1_g(M, S^1)$ is too restrictive and unusable within FEM
- Ginzburg-Landau functional is meaningful for flat or closed surfaces, and it is consistent for two-manifolds

$$\min_{f \in H^1_g(M,\mathbb{C})} \int_M |\nabla f|^2 d\mathbf{x} + \frac{1}{2\epsilon^2} \int_M (|f|^2 - 1)^2 d\mathbf{x}$$

$$E_\epsilon(\mathbf{x}^c) \approx \pi \left(\mathfrak{I}(\mathbf{x}^c)\right)^2 |\log(\epsilon)|$$

For meshing examples, see Georgiadis 2017

For further details, see talk 6B.2 Jezdimirovic

Thank you for your attention!

Any questions?

This work is funded by **ARC WAVES** 15/19-03

[*] "Ginzburg-Landau vortices", F. Bethuel et al.

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